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Abstract

In this paper, we prove new convergence results improving the ones by Chassagneux, Élie and Kharroubi [*Ann. Appl. Probab.* **22** (2012) 971–1007] for the discrete-time approximation of multidimensional obliquely reflected BSDEs. These BSDEs, arising in the study of switching problems, were considered by Hu and Tang [*Probab. Theory Related Fields* **147** (2010) 89–121] and generalized by Hamadène and Zhang [*Stochastic Process. Appl.* **120** (2010) 403–426] and Chassagneux, Élie and Kharroubi [*Electron. Commun. Probab.* **16** (2011) 120–128]. Our main result is a rate of convergence obtained in the Lipschitz setting and under the same structural conditions on the generator as the one required for the existence and uniqueness of a solution to the obliquely reflected BSDE.

Key words: BSDE with oblique reflections, discrete time approximation, switching problems.

MSC Classification (2000): 93E20, 65C99, 60H30.

1 Introduction

In this paper, we study the discrete-time approximation of the following system of reflected backward stochastic differential equations

$$\begin{cases} Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s + K_T - K_t, & 0 \leqslant t \leqslant T, \\ Y_t^\ell \geqslant \max_{j\in\mathcal{I}}\{Y_t^j - c^{\ell j}(X_t)\}, & 0 \leqslant t \leqslant T,\ \ell \in \mathcal{I}, \\ \int_0^T \Big[Y_t^\ell - \max_{j\in\mathcal{I}\setminus\{\ell\}}\{Y_t^j - c^{\ell j}(X_t)\}\Big]\,\mathrm{d}K_t^\ell = 0, & \ell \in \mathcal{I}, \end{cases} \tag{1.1}$$

where $\mathcal{I} := \{1, \ldots, d\}$, f, g and $(c^{ij})_{i,j\in\mathcal{I}}$ are Lipschitz functions and X is solution to the following forward stochastic differential equation (SDE) with Lipschitz coefficients

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s. \tag{1.2}$$

An important motivation for this study comes from economics applications, especially to energy markets. Indeed, it has been shown that the solution to the above equations allows to compute the solution of optimal switching problems which are linked to real option pricing (see e.g. [3]). This motivated a huge literature on switching problems both on the financial economics and applied mathematics sides, as pointed out in the introduction of [16]. The theoretical study of equation (1.1) has started in dimension 2 in the paper [14] and was latter extended in higher dimension in [9, 3, 22]. These studies are related to optimal switching problem and, in terms of existence and uniqueness result to (1.1), impose really strong conditions on the driver f of the BSDEs. These conditions were then weakened successively in [17, 16, 7]. It is quite important to notice that contrary to normally reflected BSDEs [13], the best existence and uniqueness result available in the literature requires structural conditions, see below, both on the driver f and the function c. To the best of our knowledge, it can be found in the paper [15].

The numerical study of (1.1) by probabilistic methods has attracted much less attention [22, 11, 8]. The first rate of convergence for a numerical scheme associated to (1.1) was proved in [7] but under quite restrictive condition on the driver f. The main goal of our work is actually to prove a rate of convergence for a discrete-time scheme to obliquely reflected BSDEs under the same conditions on f required to have existence and uniqueness to (1.1) and minimal Lipschitz condition on the function c.

As in [1, 19, 8], we first introduce a discretely reflected version of (1.1), where the reflection occurs only on a deterministic grid $\Re = \{r_0 := 0, \ldots, r_\kappa := T\}$: $Y_T^\Re = \widetilde{Y}_T^\Re := g(X_T) \in \mathcal{Q}(X_T)$, and, for $j \leqslant \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\begin{cases} \widetilde{Y}_t^\Re = Y_{r_{j+1}}^\Re + \int_t^{r_{j+1}} f(X_u, \widetilde{Y}_u^\Re, Z_u^\Re)\,\mathrm{d}u - \int_t^{r_{j+1}} Z_u^\Re\,\mathrm{d}W_u, \\ Y_t^\Re = \widetilde{Y}_t^\Re \mathbf{1}_{\{t\notin\Re\}} + \mathcal{P}(X_t, \widetilde{Y}_t^\Re)\mathbf{1}_{\{t\in\Re\}}, \end{cases} \tag{1.3}$$

where $\mathcal{P}(x,.)$ is the oblique projection operator on the closed convex domain

$$\mathcal{Q}(x) := \left\{ y \in \mathbb{R}^d | y^i \geqslant \max_{j \in \mathcal{I}} (y^j - c^{ij}(x)), \forall i \in \mathcal{I} \right\},$$

defined by

$$\mathcal{P} : (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \left(\max_{j \in \mathcal{I}} \{ y^j - c^{ij}(x) \} \right)_{1 \leqslant i \leqslant d}.$$

We denote $|\Re|$ the modulus of $\Re$ given by $|\Re| := \max_{0 \leqslant i \leqslant \kappa - 1} |r_{i+1} - r_i|$.

An important step in our study is to prove that these discretely reflected BSDEs are a good approximation of the continuously reflected ones (1.1). In section 4, we are able to control the error in terms of $|\Re|$ under minimal Lipschitz condition for the cost functions c, which is new in the literature, improving, in particular, the results of [8].

We then consider a Euler type approximation scheme associated to the BSDE (1.3) defined on a grid $\pi = \{t_0, \dots, t_n\}$ by $Y_n^{\Re,\pi} := g(X_T^\pi)$ and, for $i \in \{n-1, \dots, 0\}$,

$$\begin{cases} Z_i^{\Re,\pi} := \mathbb{E}[Y_{i+1}^{\Re,\pi} H_i \mid \mathcal{F}_{t_i}], \\ \widetilde{Y}_i^{\Re,\pi} := \mathbb{E}[Y_{i+1}^{\Re,\pi} \mid \mathcal{F}_{t_i}] + h_i f(X_{t_i}^\pi, \widetilde{Y}_i^{\Re,\pi}, Z_i^{\Re,\pi}), \\ Y_i^{\Re,\pi} := \widetilde{Y}_i^{\Re,\pi} \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(X_{t_i}^\pi, \widetilde{Y}_i^{\Re,\pi}) \mathbf{1}_{\{t_i \in \Re\}}, \end{cases} \tag{1.4}$$

where X^π is the Euler scheme associated to X, $h_i := t_{i+1} - t_i$ and weights $(H_i)_{0 \leqslant i \leqslant n-1}$ are matrices in $\mathcal{M}^{1,d}$ given by

$$(H_i)^\ell = \frac{-R}{h_i} \vee \frac{W_{t_{i+1}}^\ell - W_{t_i}^\ell}{h_i} \wedge \frac{R}{h_i}, \quad 1 \leqslant \ell \leqslant d,$$

with R a positive parameter. We denote $|\pi|$ the modulus of π given by $|\pi| := \max_{0 \leqslant i \leqslant n-1} h_i$ and we assume that we always have $\Re \subset \pi$.

To obtain our convergence results, we work, throughout this paper, under the following assumption:
(Hf)

(i) The functions $\sigma : \mathbb{R}^d \to \mathcal{M}^{d,d}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz-continuous functions.

(ii) The functions $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}^{d,d} \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}^d$ and $(c^{ij} : \mathbb{R}^d \to \mathbb{R})_{i,j \in \mathcal{I}}$ are Lipschitz-continuous functions and $f^j(x,y,z) = f^j(x,y,z^{j\cdot})$. We denote by L^Y and L^Z the Lipschitz constants of f with respect to y and z.

(iii) $g(x) \in \mathcal{Q}(x)$, for all $x \in \mathbb{R}^d$.

(iv) The cost functions $(c^{ij})_{i,j \in \mathcal{I}}$ satisfy the following structure condition

$$\begin{cases} c^{ii} = 0, & \text{for } 1 \leqslant i \leqslant d; \\ \inf_{x \in \mathbb{R}^d} c^{ij}(x) > 0, & \text{for } 1 \leqslant i,j \leqslant d \text{ with } i \neq j; \\ \inf_{x \in \mathbb{R}^d} \{ c^{ij}(x) + c^{jl}(x) - c^{il}(x) \} > 0, & \text{for } 1 \leqslant i,j \leqslant d \text{ with } i \neq j, j \neq l. \end{cases} \tag{1.5}$$

Let us emphasize here the fact that our results are obtained without any assumption on the non-degeneracy of the volatility matrix σ. We also point out that $(Hf)(ii)$ is the best condition – up to now – for existence and uniqueness to (1.1) to hold.

A fundamental result to obtain convergence for continuously reflected BSDEs is first to prove that the scheme given in (1.4) approximates efficiently discretely reflected BSDEs. This result is interesting in itself if one is only interested in the approximation of Bermudan switching problem (i.e. when the switching times are restricted to lie in the grid $\Re$). It is discussed in section 3 below and requires, in particular, the use of a new representation result for the scheme (1.4).

Combining the fact that discretely reflected BSDEs are a good approximation of continuously reflected BSDEs and that the scheme (1.4) is also a good approximation of (1.3), we obtain our new convergence result, which is the main result of this paper and is summarised in the following Theorem.

Theorem 1.1. *Let us assume that (Hf) is in force. Set R such that $L^Z R \leqslant 1$, π such that $L^Y|\pi| < 1$ and define $\alpha(|\pi|) = \log(2T/|\pi|)$. Then the following holds, for some positive constant C:*

(i) Taking $|\Re| \sim |\pi|^{1/2}$, we have

$$\sup_{0\leqslant i\leqslant n} \mathbb{E}\Big[|Y_{t_i} - \widetilde{Y}_i^{\Re,\pi}|^2 + |Y_{t_i} - Y_i^{\Re,\pi}|^2\Big] \leqslant C|\pi|^{1/2}\alpha(|\pi|).$$

(ii) Taking $|\Re| \sim |\pi|^{1/3}$, we have

$$\sup_{0\leqslant i\leqslant n} \mathbb{E}\Big[|Y_{t_i} - \widetilde{Y}_i^{\Re,\pi}|^2 + |Y_{t_i} - Y_i^{\Re,\pi}|^2\Big] \leqslant C|\pi|^{1/3}\alpha(|\pi|),$$

and

$$\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |Z_s - Z_i^{\Re,\pi}|^2 ds\right] \leqslant C|\pi|^{1/6}\sqrt{\alpha(|\pi|)}.$$

Moreover, if the cost functions c are constant, then the previous estimates remain true with $\alpha(|\pi|) := 1$.

It is important to compare the previous result with Theorem 5.4 in [8] which gives also rates of convergence for the discrete-time approximation of obliquely reflected BSDEs. Up to a slight modification of the scheme (introducing the truncation of the Brownian increments), we see that we are able to obtain the convergence rate 1/4, when the previous result, under (Hf), were only predicting a logarithmic convergence. Also, we are able to work under a minimal Lipschitz condition for the cost functions, which was not the case before.

The rest of the paper is organised as follows. In Section 2, we present preliminary results that will be useful in the rest of the paper. We discuss the representation property of obliquely reflected BSDEs in terms of auxiliary one-dimensional BSDEs. We also give new regularity results for the discretely reflected BSDEs which are key tools to obtain our convergence results. Section 3 is devoted to the study of the numerical scheme, in particular its fundamental stability property. Using this stability property and the regularity results given in Section 2, we prove a control of the error between the scheme and the discretely reflected BSDEs. Section 4 is concerned with the approximation of continuously reflected BSDEs by the discretely reflected ones. A convergence rate is obtained that allows to prove, using the result of Section 3, our main result, Theorem 1.1 above. For the reader convenience, some technical proofs are postponed in an Appendix Section.

Notations Throughout this paper we are given a finite time horizon T and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a d-dimensional standard Brownian motion $(W_t)_{t \geqslant 0}$. The filtration $(\mathcal{F}_t)_{t \leqslant T}$ is the Brownian filtration. $\mathscr{P}$ denotes the σ-algebra on $[0,T] \times \Omega$ generated by progressively measurable processes. Any element $x \in \mathbb{R}^n$ will be identified to a column vector with ith component x^i and Euclidean norm $|x|$. For $x, y \in \mathbb{R}^n$, $x.y$ denotes the scalar product of x and y. We denote by $\preccurlyeq$ the component-wise partial ordering relation on vectors. $\mathcal{M}^{n,m}$ denotes the set of real matrices with n lines and m columns. For a matrix $M \in \mathcal{M}^{n,m}$, M^{ij} is the component at row i and column j, $M^{i\cdot}$ is the ith row and $M^{\cdot j}$ the jth column.

We denote by $\mathcal{C}^{k,b}$ the set of functions with continuous and bounded derivatives up to order k. For a function $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$, we denote by $\partial_x f = (\partial_{x^1} f, \ldots, \partial_{x^n} f)$. If $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ we denote $\partial_x f$ (resp. $\partial_y f$) the derivatives with respect to the variable x (resp. y). For $g : \mathbb{R}^n \mapsto \mathbb{R}^d$, $x \to g(x)$, $\partial_x g$ is a matrix and $(\partial_x g)^{i\cdot} = \partial_x g^i$.

For ease of notation, we will sometimes write $\mathbb{E}_t[.]$ instead of $\mathbb{E}[.|\mathcal{F}_t]$, $t \in [0,T]$. Finally, for any $p \geqslant 1$, we introduce the following:

- $\mathscr{L}^p$ the set of $\mathcal{F}_T$-measurable random variables G satisfying $|G|_{\mathscr{L}^p} := \mathbb{E}[|G|^p]^{\frac{1}{p}} < +\infty$,

- $\mathscr{S}^p$ the set of càdlàg adapted processes U satisfying

$$|U|_{\mathscr{S}^p} := \mathbb{E}\left[\sup_{t \in [0,T]} |U_t|^p\right]^{\frac{1}{p}} < \infty,$$

  and $\mathscr{S}^p_c$ the subset of continuous processes in $\mathscr{S}^p$,

- $\mathscr{H}^p$ the set of progressively measurable processes V satisfying

$$|V|_{\mathscr{H}^p} := \mathbb{E}\left[\left(\int_0^T |V_t|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} < \infty,$$

- $\mathscr{K}^p$ the set of continuous non-decreasing processes in $\mathscr{S}^p$,

- $\mathscr{K}^{\Re,p}$ the set of pure jump non-decreasing processes in $\mathscr{S}^p$ with jump times in $\Re$.

In the sequel, we denote by C a constant whose value may change from line to line but which never depends on $|\pi|$ nor $|\Re|$. The notation C_α is used to stress the fact that the constant depends on some parameter α.

2 Preliminary results

In this section, we present key properties of continuously and discretely reflected BSDEs. We start by recalling the representation property in terms of "switched" BSDEs of the multidimensional systems of reflected BSDEs (1.1) or (1.3).

In a second part, we study the regularity properties of the solution to discretely reflected BSDEs in a Markovian setting. These results are key tools to obtain a convergence rate for the numerical approximation. They are new in the framework of this paper but their proofs rely on arguments that are now quite well understood.

2.1 Representation of obliquely reflected BSDEs

As mentioned in the introduction, the motivation to work on the above class of obliquely reflected BSDEs comes from the study of "switching problems" in the financial economics literature. Indeed, RBSDEs provide a characterization of the solution to these switching problems. Interestingly, the interpretation of the RBSDE in term of the solution of a "switching problem" is a key tool in our work. We now recall the link between the two objects, which takes the form of a representation theorem for the solution of the RBSDEs in terms of "switched BSDEs". This link has been established before, see e.g. [17]. We state it here in a generic framework as this will be useful latter on.

We consider a matrix valued process $C = (C^{ij})_{1\leqslant i,j\leqslant n}$ such that C^{ij} belongs to $\mathscr{S}^2$ for $i,j \in \mathcal{I}$ and satisfies the structure condition

$$\begin{cases} C^{ii}_t = 0, & \text{for } 1 \leqslant i \leqslant d \text{ and } 0 \leqslant t \leqslant T; \\ \inf_{t\in[0,T]} C^{ij}_t \geqslant \varepsilon > 0, & \text{for } 1 \leqslant i,j \leqslant d \text{ with } i \neq j; \\ \inf_{t\in[0,T]}\{C^{ij}_t + C^{jl}_t - C^{il}_t\} > 0, & \text{for } 1 \leqslant i,j \leqslant d \text{ with } i \neq j, j \neq l. \end{cases} \tag{2.1}$$

We introduce a random closed convex set family associated to C:

$$\mathcal{Q}_t := \left\{ y \in \mathbb{R}^d | y^i \geqslant \max_{j\in\mathcal{I}} (y^j - C^{ij}_t), 1 \leqslant i \leqslant d \right\}, \quad 0 \leqslant t \leqslant T,$$

and the oblique projection operator onto $\mathcal{Q}_t$, denoted $\mathcal{P}_t$ and defined by

$$\mathcal{P}_t : y \in \mathbb{R}^d \mapsto \left(\max_{j\in\mathcal{I}} \{y^j - C^{ij}_t\} \right)_{1\leqslant i\leqslant d}. \tag{2.2}$$

Remark 2.1. *It follows from the structure condition* (2.1) *that $\mathcal{P}_t$ is increasing with respect to the partial ordering relation $\leqslant$.*

A switching strategy a is a nondecreasing sequence of stopping times $(\theta_j)_{j\in\mathbb{N}}$, combined with a sequence of random variables $(\alpha_j)_{j\in\mathbb{N}}$ valued in $\mathcal{I}$, such that α_j is $\mathcal{F}_{\theta_j}-$measurable, for any $j \in \mathbb{N}$. We denote by $\mathscr{A}$ the set of such strategies. For $a = (\theta_j, \alpha_j)_{j\in\mathbb{N}} \in \mathscr{A}$, we introduce $\mathcal{N}^a$ the (random) number of switches before T:

$$\mathcal{N}^a = \#\{k \in \mathbb{N}^* \ : \ \theta_k \leqslant T\} . \tag{2.3}$$

To any switching strategy $a = (\theta_j, \alpha_j)_{j\in\mathbb{N}} \in \mathscr{A}$, we associate the current state process $(a_t)_{t\in[0,T]}$ and the cumulative cost process $(\mathcal{A}^a_t)_{t\in[0,T]}$ defined respectively by

$$a_t \ := \ \alpha_0 \mathbf{1}_{\{0\leqslant t<\theta_0\}} + \sum_{j=1}^{\mathcal{N}^a} \alpha_{j-1}\mathbf{1}_{\{\theta_{j-1}\leqslant t<\theta_j\}} \ \text{ and } \ \mathcal{A}^a_t \ := \ \sum_{j=1}^{\mathcal{N}^a} C^{\alpha_{j-1}\alpha_j}_{\theta_j}\mathbf{1}_{\{\theta_j\leqslant t\leqslant T\}} ,$$

for $0 \leqslant t \leqslant T$.

Remark 2.2. *(i) The sequence of stopping times is only supposed to be non-decreasing, but the assumptions on the cost processes* (2.1) *imply that any reasonable strategy uses a sequence of increasing stopping times. This is specially the case for the optimal strategies.*

(ii) Note that the cumulative cost process will keep track of all the switching times, even the instantaneous ones; whereas the state process will keep track of the last state when instantaneous switches occur.

For $(t, i) \in [0, T] \times \mathcal{I}$, the set $\mathscr{A}_{t,i}$ of admissible strategies starting from state i at time t is defined by

$$\mathscr{A}_{t,i} = \{a = (\theta_j, \alpha_j)_j \in \mathscr{A} \ |\theta_0 = t, \ \alpha_0 = i, \ \mathbb{E}\big[|\mathcal{A}^a_T|^2\big] < \infty\} ,$$

similarly we introduce $\mathscr{A}^{\Re}_{t,i}$ the restriction to $\Re-$admissible strategies

$$\mathscr{A}^{\Re}_{t,i} := \{ \ a = (\theta_j, \alpha_j)_{j\in\mathbb{N}} \in \mathscr{A}_{t,i} \quad | \quad \theta_j \in \Re \ , \ \forall j \leqslant \mathcal{N}^a \ \} ,$$

and denote $\mathscr{A}^{\Re} := \bigcup_{i\leqslant d} \mathscr{A}^{\Re}_{0,i}$.

For a strategy $a \in \mathscr{A}_{t,\ell}$, we introduce the one-dimensional *switched BSDE* whose solution $(\mathcal{U}^a, \mathcal{V}^a)$ satisfies

$$\mathcal{U}^a_t = \xi^{a_T} + \int_t^T F^{a_s}(s, \mathcal{V}^a_s)\mathrm{d}s - \int_t^T \mathcal{V}^a_s \mathrm{d}W_s - \mathcal{A}^a_T + \mathcal{A}^a_t \tag{2.4}$$

where the terminal condition ξ, the random costs process and the random driver F satisfies following assumptions, for some $p \geqslant 2$:

(HF_p)

(i) $F : \Omega \times [0, T] \times \mathcal{M}^{d,d} \to \mathbb{R}^d$ is $\mathscr{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathcal{M}^{d,d})$-measurable,

(ii) $F^j(\cdot, z) = F^j(\cdot, z^{j\cdot})$ for all $j \in \mathcal{I}$,

(iii) $|F(s,z) - F(s,z')| \leqslant C|z - z'|$ for all $s \in [0,T]$, $z, z' \in \mathcal{M}^{d,d}$,

(iv) ξ is $\mathcal{F}_T$-measurable and is valued in $\mathcal{Q}_T$,

(v) $\mathbb{E}\left[|\xi|^p + \int_0^T |F(s,0)|^p\,ds\right] \leqslant C_p$.

We now define multidimensional processes $\bar{\mathcal{Y}}$ and $\bar{\mathcal{Y}}^{\Re}$ as follows, for $\ell \in \{1,\dots,d\}$

$$(\bar{\mathcal{Y}}_t)^\ell := \operatorname*{ess\,sup}_{a \in \mathscr{A}_{t,\ell}} \mathcal{U}_t^a \quad \text{and} \quad (\bar{\mathcal{Y}}_t^{\Re})^\ell := \operatorname*{ess\,sup}_{a \in \mathscr{A}_{t,\ell}^{\Re}} \mathcal{U}_t^a\,.$$

The process $\mathcal{Y}$ represents the optimal value that can be obtained from the switched BSDEs following strategies in $\mathscr{A}$. The process $\bar{\mathcal{Y}}^{\Re}$ can be seen as a "Bermudan" version of it i.e. when the switching times are restricted to lie in $\Re$. Both processes enjoy a representation in terms of reflected BSDEs, the main difference lying into the reflecting process that for the latter will be a pure jump process with jump times in $\Re$.

Let $(\mathcal{Y}, \mathcal{Z}, \mathcal{K})$ be the solution to the following BSDE

$$\begin{cases} \mathcal{Y}_t^\ell = \xi^\ell + \displaystyle\int_t^T F^\ell(s,\mathcal{Z}_s)\,\mathrm{d}s - \int_t^T \mathcal{Z}_s^\ell\,\mathrm{d}W_s + \mathcal{K}_T^\ell - \mathcal{K}_t^\ell, & 0 \leqslant t \leqslant T,\ \ \ell \in \mathcal{I}, \\ \mathcal{Y}_t^\ell \geqslant \max\limits_{j \in \mathcal{I}}\{\mathcal{Y}_t^j - C_t^{\ell j}\}, & 0 \leqslant t \leqslant T,\ \ \ell \in \mathcal{I}, \\ \displaystyle\int_0^T \Big[\mathcal{Y}_t^\ell - \max_{j \in \mathcal{I}\setminus\{\ell\}}\{\mathcal{Y}_t^j - C_t^{\ell j}\}\Big]\,\mathrm{d}\mathcal{K}_t^\ell = 0, & \ell \in \mathcal{I}, \end{cases} \tag{2.5}$$

and $(\tilde{\mathcal{Y}}^{\Re}, \mathcal{Y}^{\Re}, \mathcal{Z}^{\Re}, \mathcal{K}^{\Re})$ with $\mathcal{Y}_t^{\Re} = \tilde{\mathcal{Y}}_{t-}^{\Re}$, $t \in (0,T]$ be the solution of following discretely reflected BSDEs,

$$\begin{cases} \tilde{\mathcal{Y}}_t^{\Re} = \xi + \displaystyle\int_t^T F(s,\mathcal{Z}_s^{\Re})\,\mathrm{d}s - \int_t^T \mathcal{Z}_s^{\Re}\,\mathrm{d}W_s + \mathcal{K}_T^{\Re} - \mathcal{K}_t^{\Re}, & 0 \leqslant t \leqslant T, \\ \mathcal{Y}_r^{\Re} \in \mathcal{Q}_r, & r \in \Re, \\ \displaystyle\int_0^T \Big[(\mathcal{Y}_t^{\Re})^\ell - \max_{j \in \mathcal{I}\setminus\{\ell\}}\{(\mathcal{Y}_t^{\Re})^\ell - C_t^{\ell j}\}\Big]\,\mathrm{d}(\mathcal{K}_t^{\Re})^\ell = 0, & \ell \in \mathcal{I}, \end{cases} \tag{2.6}$$

Existence and uniqueness of a solution for equation (2.5) has been addressed in [17, 16] and in [8] (Proposition 2.1) for equation (2.6). For the reader convenience we recall here these results.

Proposition 2.1. *Assume that (HF_p) holds for some $p \geqslant 2$. There exists a unique solution $(\mathcal{Y}, \mathcal{Z}, \mathcal{K}) \in \mathscr{S}_c^2 \times \mathscr{H}^2 \times \mathscr{K}^2$ to (2.5) and a unique solution $(\tilde{\mathcal{Y}}^{\Re}, \mathcal{Y}^{\Re}, \mathcal{Z}^{\Re}, \mathcal{K}^{\Re})$ with $(\tilde{\mathcal{Y}}^{\Re}, \mathcal{Z}^{\Re}, \mathcal{K}^{\Re}) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^{\Re,2}$ to (2.6). They also satisfy*

$$|\mathcal{Y}|_{\mathscr{S}^p} + |\mathcal{Z}|_{\mathscr{H}^p} + |\mathcal{K}_T|_{\mathscr{L}^p} \leqslant C_p \ \ \textit{and} \ \ |\tilde{\mathcal{Y}}^{\Re}|_{\mathscr{S}^p} + |\mathcal{Z}^{\Re}|_{\mathscr{H}^p} + |\mathcal{K}_T^{\Re}|_{\mathscr{L}^p} \leqslant C_p.$$

Gathering Proposition 3.2 in [7] and Theorem 2.1 in [8], we have the following key representation result.

Proposition 2.2. *Assume that* (HF_2) *is in force. The following hold:*

(i) for all $\ell \in \{1, \dots, d\}$, $t \in [0, T]$,

$$(\mathcal{Y}_t)^\ell = (\bar{\mathcal{Y}}_t)^\ell = \mathcal{U}_t^{\bar{a}} \quad \text{and} \quad (\tilde{\mathcal{Y}}_t^{\Re})^\ell = (\bar{\mathcal{Y}}_t^{\Re})^\ell = \mathcal{U}_t^{\bar{a}^{\Re}}$$

for some $\bar{a} \in \mathscr{A}_{t,\ell}$ *and* $\bar{a}^{\Re} \in \mathscr{A}_{t,\ell}^{\Re}$.

(ii) The strategy $\bar{a} = (\bar{\theta}_j, \bar{\alpha}_j)_{j \geqslant 0}$ *can be defined recursively by* $(\bar{\theta}_0, \bar{\alpha}_0) := (t, \ell)$ *and, for* $j \geqslant 1$,

$$\bar{\theta}_j := \inf \left\{ s \in [\bar{\theta}_{j-1}, T] \,\middle|\, (\tilde{\mathcal{Y}}_s)^{\bar{\alpha}_{j-1}} \leqslant \max_{k \neq \bar{\alpha}_{j-1}} \{ (\tilde{\mathcal{Y}}_s)^k - C_s^{\bar{\alpha}_{j-1} k} \} \right\},$$

$$\bar{\alpha}_j := \min \left\{ \ell \neq \bar{\alpha}_{j-1} \,\middle|\, (\tilde{\mathcal{Y}}_{\bar{\alpha}_j})^\ell - C_{\bar{\theta}_j}^{\bar{\alpha}_{j-1}\ell} = \max_{k \neq \bar{\alpha}_{j-1}} \{ (\tilde{\mathcal{Y}}_{\bar{\theta}_j})^k - C_{\bar{\theta}_j}^{\bar{\alpha}_{j-1} k} \} \right\}.$$

(iii) The strategy $\bar{a}^{\Re} = (\bar{\theta}_j^{\Re}, \bar{\alpha}_j^{\Re})_{j \geqslant 0}$ *can be defined recursively by* $(\bar{\theta}_0^{\Re}, \bar{\alpha}_0^{\Re}) := (t, \ell)$ *and, for* $j \geqslant 1$,

$$\bar{\theta}_j^{\Re} := \inf \left\{ s \in [\bar{\theta}_{j-1}^{\Re}, T] \cap \Re \,\middle|\, (\tilde{\mathcal{Y}}_s^{\Re})^{\bar{\alpha}_{j-1}^{\Re}} \leqslant \max_{k \neq \bar{\alpha}_{j-1}^{\Re}} \{ (\tilde{\mathcal{Y}}_s^{\Re})^k - C_s^{\bar{\alpha}_{j-1}^{\Re} k} \} \right\},$$

$$\bar{\alpha}_j^{\Re} := \min \left\{ \ell \neq \bar{\alpha}_{j-1}^{\Re} \,\middle|\, (\tilde{\mathcal{Y}}_{\bar{\theta}_j^{\Re}}^{\Re})^\ell - C_{\bar{\theta}_j^{\Re}}^{\bar{\alpha}_{j-1}^{\Re}\ell} = \max_{k \neq \bar{\alpha}_{j-1}^{\Re}} \{ (\tilde{\mathcal{Y}}_{\bar{\theta}_j^{\Re}}^{\Re})^k - C_{\bar{\theta}_j^{\Re}}^{\bar{\alpha}_{j-1}^{\Re} k} \} \right\}.$$

Remark 2.3. *If* $\tilde{Y}_t^\ell \notin \mathcal{Q}_t$ *then there is an instantaneous jump, i.e.* $\bar{\theta}_1 = t$. *In the same way, if* $t \in \Re$ *and* $(\tilde{Y}_t^{\Re})^\ell \notin \mathcal{Q}_t$ *then* $\bar{\theta}_1^{\Re} = t$.

2.2 Discretely obliquely reflected BSDEs in a Markovian setting

We will now study the discretely obliquely reflected BSDEs (2.6) in a Markovian setting, namely the solution to (1.3). We will in particular prove regularity results for this process. The main difference with Section 3 in [8] comes from the assumption on f, in particular the full dependence in the y-variable, recall (Hf)(ii).

Let us recall that under assumption (Hf)(i), there exists a unique strong solution to the SDE (1.2) which satisfies

$$\mathbb{E}_t\left[\sup_{s \in [t,T]} |X_s|^p \right] \leqslant C_p(1 + |X_t|^p), \quad \text{for all } p \geqslant 2, \ t \in [0, T]. \tag{2.7}$$

2.2.1 Basic properties

The following proposition gives some usefull estimates on the solution to (1.3). Its proof is postponed to the Appendix.

Proposition 2.3. *Assume that* (Hf) *is in force. There exists a unique solution* $(\tilde{Y}^{\Re}, Y^{\Re}, Z^{\Re}) \in \mathscr{S}^2 \times \mathscr{S}^2 \times \mathscr{H}^2$ *to* (1.3) *and it satisfies, for all* $p \geqslant 2$,

$$|\tilde{Y}^{\Re}|_{\mathscr{S}^p} + |Z^{\Re}|_{\mathscr{H}^p} + |K_T^{\Re}|_{\mathscr{L}^p} \leqslant C_p.$$

We now precise the results of Proposition 2.2, in the setting of this section. In particular, we describe the optimal strategy and some of its properties that will be useful in the sequel.

Corollary 2.1. *(i) The following equalities hold, for all* $\ell \in \{1, \dots, d\}$, $t \in [0, T]$,

$$(\tilde{Y}_t^{\Re})^{\ell} = \operatorname*{ess\,sup}_{a \in \mathscr{A}_{t,\ell}^{\Re}} U_t^{\Re,a} = U_t^{\Re,\bar{a}^{\Re}} \quad \textit{for some } \bar{a}^{\Re} \in \mathscr{A}_{t,\ell}^{\Re} ,$$

where $(U^{\Re,a}, V^{\Re,a}, N^{\Re,a})$ *is solution of the switched BSDE* (2.4) *with random driver* $F(s, z) := f(s, X_s, \tilde{Y}_s^{\Re}, z)$ *for* $(s, z) \in [0, T] \times \mathcal{M}^{d,d}$, *terminal condition* $\xi := g(X_T)$ *and costs* $C_s^{ij} = c^{ij}(X_s)$.

(ii) The optimal strategy $\bar{a}^{\Re} = (\theta_j, \alpha_j)_{j \geqslant 0}$ *can be defined recursively by* $(\theta_0, \alpha_0) := (t, \ell)$ *and, for* $j \geqslant 1$,

$$\theta_j := \inf \left\{ s \in [\theta_{j-1}, T] \cap \Re \;\middle|\; (\tilde{Y}_s^{\Re})^{\alpha_{j-1}} \leqslant \max_{k \neq \alpha_{j-1}} \left\{ (\tilde{Y}_s^{\Re})^k - c^{\alpha_{j-1}k}(X_s) \right\} \right\} ,$$

$$\alpha_j := \min \left\{ q \neq \alpha_{j-1} \;\middle|\; (\tilde{Y}_{\theta_j}^{\Re})^q - c^{\alpha_{j-1}q}(X_{\theta_j}) = \max_{k \neq \alpha_{j-1}} \left\{ (\tilde{Y}_{\theta_j}^{\Re})^k - c^{\alpha_{j-1}k}(X_{\theta_j}) \right\} \right\} .$$

(iii) Moreover, for all $\ell \in \{1, \dots, d\}$, $t \in [0, T]$, *the optimal strategy* $\bar{a}^{\Re} \in \mathscr{A}_{t,\ell}^{\Re}$ *satisfies*

$$\mathbb{E}_t\left[\sup_{s \in [t,T]} \left| U_s^{\Re,\bar{a}^{\Re}} \right|^p + \left(\int_t^T \left| V_s^{\Re,\bar{a}^{\Re}} \right|^2 ds \right)^{p/2} + \left| A_T^{\Re,\bar{a}^{\Re}} \right|^p + \left| N^{\Re,\bar{a}^{\Re}} \right|^p \right] \leqslant C_p(1 + |X_t|^p). \tag{2.8}$$

Proof. Thanks to Proposition 2.3, we can apply Proposition 2.2 with the random driver F, the terminal condition ξ and costs C_s^{ij} defined above, which gives us the representation result. The first estimate in (2.8) is a direct application of this representation result and Proposition 2.3. Other estimates in (2.8) are obtained by using standard arguments for BSDEs combined with the estimate (A.1), see proof of Proposition 2.2 in [8] for details. □

2.2.2 Fine estimates on $(Y^{\Re}, \tilde{Y}^{\Re}, Z^{\Re})$

In this section, we prove regularity results on the solution $(Y^{\Re}, \tilde{Y}^{\Re}, Z^{\Re})$ of the discretely reflected BSDEs. To do that, we will use techniques already exposed in [1, 5, 8], based essentially on a representation of $Z^{\Re}$, obtained by using Malliavin Calculus. For a general presentation of Malliavin Calculus, we refer to [20]. We now introduce some notations and recall some known results on Malliavin differentiability of SDEs solution.

We will work under the following assumption.
(Hr) The coefficients b, σ, g f, and $(c^{ij})_{i,j}$ are $C^{1,b}$ in all their variables, with the Lipschitz constants dominated by L.
This assumption is classically relieved using a kernel regularisation argument, see e.g. the proofs of Proposition 4.2 in [5] or Proposition 3.3 in [1].

We denote by $\mathbb{D}^{1,2}$ the set of random variables G which are differentiable in the Malliavin sense and such that $\|G\|^2_{\mathbb{D}^{1,2}} := \|G\|^2_{\mathscr{L}^2} + \int_0^T \|D_t G\|^2_{\mathscr{L}^2} dt < \infty$, where $D_t G$ denotes the Malliavin derivative of G at time $t \leqslant T$. After possibly passing to a suitable version, an adapted process belongs to the subspace $\mathcal{L}^{1,2}_a$ of $\mathscr{H}^2$ whenever $V_s \in \mathbb{D}^{1,2}$ for all $s \leqslant T$ and $\|V\|^2_{\mathcal{L}^{1,2}_a} := \|V\|^2_{\mathscr{H}^2} + \int_0^T \|D_t V\|^2_{\mathscr{H}^2} dt < \infty$.

Remark 2.4. Under (Hr), the solution of (1.2) is Malliavin differentiable and its derivative satisfies

$$\|\sup_{s \leqslant T} |D_s X|\|_{\mathscr{L}^p} < \infty \text{ and } \mathbb{E}_r\Big[\sup_{r \leqslant s \leqslant T} |D_u X_s|^p\Big] \leqslant C(1 + |X_r|^p), \quad u \leqslant r \leqslant T. \qquad (2.9)$$

Moreover, we have

$$\sup_{s \leqslant u} \|D_s X_t - D_s X_u\|_{\mathscr{L}^p} + \|\sup_{t \leqslant s \leqslant T} |D_t X_s - D_u X_s|\,\|_{\mathscr{L}^p} \leqslant C^p_L |t-u|^{1/2}, \qquad (2.10)$$

for any $0 \leqslant u \leqslant t \leqslant T$.

Malliavin derivatives of $(Y^{\Re}, \tilde{Y}^{\Re}, Z^{\Re})$. We now study the Malliavin differentiability of $(Y^{\Re}, \tilde{Y}^{\Re}, Z^{\Re})$. The techniques used are classical by now, see [1, 5]. In this paragraph, we will follow the presentation of [8]. Once again, the main difference with this paper is the assumption (Hf) made on the driver f. In the setting of [8], f has to satisfy $f^i(x,y,z) = f^i(x,y^i,z^i)$ whereas (Hf) does not impose such restriction on the y variable. This implies that the representation of Z, see Corollary 2.2 below, is slightly more complicated. Namely, it contains the term $D\tilde{Y}$, compare to Proposition 3.2 in [8]. To obtain the regularity results on $(Y^{\Re}, \tilde{Y}^{\Re}, Z^{\Re})$, we need thus to prove estimates on $D\tilde{Y}$, which is the main result of the next Proposition.

Proposition 2.4. *Under (Hf)-(Hr), $(\tilde{Y}^{\Re}, Z^{\Re})$ is Malliavin differentiable and its deriva-*

tive satisfies, for all $r \in [0,T]$, $u \leqslant r$, $i \in \mathcal{I}$,

$$
\begin{aligned}
D_u(\widetilde{Y}^{\Re}_r)^i = \mathbb{E}_r\Bigg[&\partial_x g^{a_T}(X_T) D_u X_T + \int_r^T \partial_x f^{a_s}(\Theta^{\Re}_s) D_u X_s \mathrm{d}s \\
&+ \int_r^T \partial_y f^{a_s}(\Theta^{\Re}_s) D_u \widetilde{Y}^{\Re}_s \mathrm{d}s + \int_r^T \sum_{\ell=1}^{d} \partial_{z^{a_s \ell}} f^{a_s}(\Theta^{\Re}_s) D_u (Z^{\Re}_s)^{a_s \ell} \mathrm{d}s \\
&- \sum_{j=1}^{N^a} \partial_x c^{\alpha_{j-1}\alpha_j}(X_{\theta_j}) D_u X_{\theta_j}\Bigg]
\end{aligned}
\tag{2.11}
$$

where $a := \bar{a}^{\Re}$ is the optimal strategy associated with the representation in terms of switched BSDEs, recall Corollary 2.1, and $\Theta^{\Re} := (X, \tilde{Y}^{\Re}, Z^{\Re})$. Moreover, the following estimates hold true: for all $r \in [0,T]$, $0 \leqslant u \leqslant r$, $0 \leqslant v \leqslant r$,

$$
|D_u \tilde{Y}_r|^2 \leqslant C_L(1 + |X_r|^2) \tag{2.12}
$$

and

$$
|D_u \tilde{Y}_r - D_v \tilde{Y}_r|^2 \leqslant C_L(1 + |X_r|) \mathbb{E}_r\left[\sup_{r \leqslant s \leqslant T} |D_u X_s - D_v X_s|^4\right]^{\frac{1}{2}} . \tag{2.13}
$$

Proof.

Let $G \in \mathbb{D}^{1,2}(\mathbb{R}^d)$. Since X belongs to $\mathcal{L}^{1,2}_a$ under (Hr), and $\mathcal{P}$ is a Lipschitz continuous function, we deduce that $\mathcal{P}(X_t, G) \in \mathbb{D}^{1,2}(\mathbb{R}^d)$. Using Lemma 5.1 in [1], we compute

$$
\begin{aligned}
&D_s(\mathcal{P}(X_t, G))^i = \\
&\sum_{j=1}^{d} (D_s G^j - D_s c_{ij}(X_t)) \mathbf{1}_{G^j - c^{ij}(X_t) > \max_{\ell < j}(G^\ell - c^{i\ell}(X_t))} \mathbf{1}_{G^j - c^{ij}(X_t) \geqslant \max_{\ell > j}(G^\ell - c^{i\ell}(X_t))}.
\end{aligned}
\tag{2.14}
$$

Combining (2.14), Proposition 5.3 in [10] and an induction argument, we obtain that $(Y^{\Re}, \widetilde{Y}^{\Re}, Z^{\Re})$ is Malliavin differentiable and that a version of $(D_u \widetilde{Y}^{\Re}, D_u Z^{\Re})$ is given by, for all $i \in \mathcal{I}$, $t \in [0,T]$, $0 \leqslant u \leqslant t$,

$$
\begin{aligned}
D_u(\widetilde{Y}^{\Re}_t)^i = &D_u(Y^{\Re}_{r_{j+1}})^i - \sum_{k=1}^{d} \int_t^{r_{j+1}} D_u(Z^{\Re}_s)^{ik} \mathrm{d}W^k_s + \int_t^{r_{j+1}} \partial_x f^i(\Theta^{\Re}_s) D_u X_s \mathrm{d}s \\
&+ \int_t^{r_{j+1}} \partial_y f^i(\Theta^{\Re}_s) D_u \widetilde{Y}^{\Re}_s \mathrm{d}s + \int_t^{r_{j+1}} \sum_{\ell=1}^{d} \partial_{z^{i\ell}} f^i(\Theta^{\Re}_s) D_u (Z^{\Re})^{i\ell}_s \mathrm{d}s
\end{aligned}
\tag{2.15}
$$

recall (Hf).

Now, we consider the optimal strategy $a := \bar{a}^{\Re}$ defined in Corollary 2.1 (ii) above and fix $j < \kappa$. Observing that the process a is constant on the interval $[\theta_j, \theta_{j+1})$, we deduce from (2.15)

$$D_u(\widetilde{Y}^{\Re}_t)^{\alpha_j} = D_u(Y^{\Re}_{\theta_{j+1}})^{\alpha_j} - \sum_{k=1}^{d}\int_t^{\theta_{j+1}} D_u(Z^{\Re}_s)^{\alpha_j k}\mathrm{d}W^k_s + \int_t^{\theta_{j+1}} \partial_x f^{\alpha_j}(\Theta^{\Re}_s)D_u X_s \mathrm{d}s \tag{2.16}$$

$$+\int_t^{\theta_{j+1}} \partial_y f^{\alpha_j}(\Theta^{\Re}_s)D_u\widetilde{Y}^{\Re}_s\mathrm{d}s + \int_t^{\theta_{j+1}}\sum_{\ell=1}^{d}\partial_{z^{\alpha_j \ell}} f^{\alpha_j}(\Theta^{\Re}_s)D_u(Z^{\Re})^{\alpha_j \ell}_s\mathrm{d}s$$

for $t \in [\theta_j, \theta_{j+1}]$ and $0 \leqslant u \leqslant t$. Combining (2.14) and the definition of a given in Corollary 2.1 (ii), we compute, for $u \leqslant \theta_{j+1}$ and $j < \kappa$,

$$D_u(Y^{\Re}_{\theta_{j+1}})^{\alpha_j} = D_u(\widetilde{Y}^{\Re}_{\theta_{j+1}})^{\alpha_{j+1}} - \partial_x c^{\alpha_j\alpha_{j+1}}(X_{\theta_{j+1}})D_u X_{\theta_{j+1}}.$$

Inserting the previous equality into (2.16) and summing up over j we obtain, for all $t \leqslant r \leqslant T$,

$$\begin{aligned} D_u(\widetilde{Y}^{\Re}_r)^i =& \partial_x g^{a_T}(X_T)D_u X_T - \int_r^T \sum_{k=1}^{d} D_u(Z^{\Re}_s)^{a_s k}\mathrm{d}W_s + \int_r^T \partial_x f^{a_s}(\Theta^{\Re}_s)D_u X_s\mathrm{d}s \\ &+\int_r^T \partial_y f^{a_s}(\Theta^{\Re}_s)D_u\widetilde{Y}^{\Re}_s\mathrm{d}s + \int_r^T \sum_{\ell=1}^{d}\partial_{z^{a_s \ell}} f^{a_s}(\Theta^{\Re}_s)D_u(Z^{\Re}_s)^{a_s\ell}\mathrm{d}s \\ &-\sum_{j=1}^{N^a}\partial_x c^{\alpha_{j-1}\alpha_j}(X_{\theta_j})(D_u X)_{\theta_j}\,. \end{aligned} \tag{2.17}$$

Taking conditional expectation on both sides of the previous equality proves (2.11). Moreover, we are in the framework of section A.2 in the Appendix by setting $\mathfrak{Y} = D_u Y$ and $\mathfrak{X} = D_u X$. Condition (A.2) is satisfied here by $N^{\bar{a}^{\Re}}$ with $\beta := C_L(1+|X|)$, recall (2.8). Using Proposition A.1 and (2.9), we then obtain (2.12).

From equation (2.17), we easily deduce the dynamics of $D_u(\widetilde{Y}^{\Re}) - D_v(\widetilde{Y}^{\Re})$, which leads, using again Proposition A.1, to (2.13). □

The representation result for $Z^{\Re}$ is then an easy consequence of the previous proposition.

Corollary 2.2. *Under (Hf)-(Hr) the following representation holds true,*

$$\begin{aligned} Z^{\Re}_t = \mathbb{E}_t\Big[& \partial_x g^{a_T}(X_T)\Lambda^a_{t,T}D_t X_T - \sum_{j=1}^{N^a}\partial_x c^{\alpha_{j-1}\alpha_j}(X_{\theta_j})\Lambda^a_{t,\theta_j}D_t X_{\theta_j} \\ &+\int_t^T\left(\partial_x f^{a_s}(\Theta^{\Re}_s)\Lambda^a_{t,s}D_t X_s + \partial_y f^{a_s}(\Theta^{\Re}_s)\Lambda^a_{t,s}D_t\widetilde{Y}^{\Re}_s\right)\mathrm{d}s\Big]\,, \end{aligned} \tag{2.18}$$

where $a := \bar{a}^{\Re}$ is the optimal strategy associated with the representation in terms of switched BSDEs, recall Corollary 2.1, and for $\ell \in \mathcal{I}$,

$$\Lambda^a_{t,s} := \exp\left(\int_t^s \partial_{z^{a_r\cdot}} f^{a_r}(\Theta^{\Re}_u)\mathrm{d}W_r - \frac{1}{2}\int_t^s |\partial_{z^{a_r\cdot}} f^{a_r}(\Theta^{\Re}_u)|^2\mathrm{d}r\right). \tag{2.19}$$

Moreover, under (Hf), we have

$$\left|Z_t^{\Re}\right| \leqslant \bar{L}(1+|X_t|), \quad \textit{for all } t \in [0,T], \tag{2.20}$$

for some positive constant $\bar{L}$ that does not depend on the grid $\Re$.

Proof. 1. A version of $Z^{\Re}$ is given by $(D_t\tilde{Y}_t^{\Re})_{0\leqslant t\leqslant T}$. The expression of D_tY is obtained directly by applying Itô's formula, recall (2.11).

2. Under (Hf)-(Hr) the estimate (2.20) follows from (2.12) and (2.9). Under (Hf), we can obtain the result by a standard kernel regularisation argument. □

Regularity of $(Y^{\Re}, Z^{\Re})$. With the above results at hand, the study of the regularity of $(Y^{\Re}, Z^{\Re})$ follows from "classical" arguments, see e.g. [5, 8]. For sake of completeness, we reproduce them below.

We consider a grid $\pi := \{t_0 = 0, \dots, t_n = T\}$ on the time interval $[0,T]$, with modulus $|\pi| := \max_{0\leqslant i\leqslant n-1} |t_{i+1}-t_i|$, such that $\Re \subset \pi$.

We need to control the following quantities, representing the $\mathscr{H}^2$-regularity of $(\tilde{Y}, Z)$:

$$\mathbb{E}\left[\int_0^T |\tilde{Y}_t^{\Re} - \tilde{Y}_{\pi(t)}^{\Re}|^2 \mathrm{d}t\right] \quad \text{and} \quad \mathbb{E}\left[\int_0^T |Z_t^{\Re} - \bar{Z}_{\pi(t)}^{\Re}|^2 \mathrm{d}t\right], \tag{2.21}$$

where $\pi(t) := \sup\{t_i \in \pi\,;\ t_i \leqslant t\}$ is defined on $[0,T]$ as the projection to the closest previous grid point of π and

$$\bar{Z}_{t_i}^{\Re} := \frac{1}{t_{i+1}-t_i}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s^{\Re}\mathrm{d}s \,|\, \mathcal{F}_{t_i}\right], \quad i \in \{0,\dots,n-1\}. \tag{2.22}$$

Remark 2.5. Observe that $(\bar{Z}_s^{\Re})_{s\leqslant T} := (\bar{Z}_{\pi(s)}^{\Re})_{s\leqslant T}$ interprets as the best $\mathscr{H}^2$-approximation of the process $Z^{\Re}$ by adapted processes which are constant on each interval $[t_i, t_{i+1})$, for all $i < n$.

The first result is the regularity of the Y-component, which is a direct consequence of the bound (2.20).

Proposition 2.5. *Under (Hf), the following holds*

$$\sup_{t\in[0,T]} \mathbb{E}\left[|\tilde{Y}_t^{\Re} - \tilde{Y}_{\pi(t)}^{\Re}|^2\right] \leqslant C_L|\pi|.$$

Proof. We first observe that, for all $0 \leqslant t \leqslant T$,

$$\begin{aligned}
\mathbb{E}\left[|\tilde{Y}_t^{\Re} - \tilde{Y}_{\pi(t)}^{\Re}|^2\right] &\leqslant \mathbb{E}\left[\left|\int_{\pi(t)}^t f(X_s, \tilde{Y}_s^{\Re}, Z_s^{\Re})\mathrm{d}s + \int_{\pi(t)}^t Z_s^{\Re}\mathrm{d}W_s\right|^2\right], \\
&\leqslant C_L\mathbb{E}\left[\int_{\pi(t)}^t \left(1+|X_s|^2+|\tilde{Y}_s^{\Re}|^2+|\tilde{Z}_s^{\Re}|^2\right)\mathrm{d}s\right]. \\
&\leqslant C_L\mathbb{E}\left[|\pi| + \int_{\pi(t)}^t |\tilde{Z}_s^{\Re}|^2\mathrm{d}s\right]
\end{aligned} \tag{2.23}$$

where we used (2.7) and Proposition 2.3. From (2.20), we easily get $\mathbb{E}\Big[\int_{\pi(t)}^{t} \big|Z_t^{\Re}\big|^2 \,\mathrm{d}t\Big] \leqslant C_L|\pi|$. Inserting the previous inequality into (2.23) concludes the proof of this Proposition. □

The following Proposition gives us the regularity of $Z^{\Re}$. Its proof is postponed to the Appendix.

Proposition 2.6. *Under (Hf), the following holds*

$$\mathbb{E}\Big[\int_0^T |Z_t^{\Re} - \bar{Z}_t^{\Re}|^2 \mathrm{d}t\Big] \leqslant C_L \left(|\pi|^{\frac{1}{2}} + \kappa|\pi|\right).$$

3 Study of the discrete-time approximation

The aim of this section is to obtain a control on the error between the obliquely reflected backward scheme (1.4) and the discretely obliquely reflected BSDE (1.3). This is the purpose of Theorem 3.1 in subsection 3.4 below. In order to prove this key result, we start by interpreting the scheme in terms of the solution of a switching problem in subsection 3.2. We then use this representation to obtain a general stability property for the scheme in subsection 3.3. Subsection 3.1 is devoted to preliminary definition and propositions.

3.1 Definition and first estimates

Given a grid π of the interval $[0,T]$, we first consider an obliquely reflected backward scheme with a random generator and a random cost process C^π. For $t \in [0,T]$, we denote by $\mathcal{Q}_t^\pi$ the random closed convex set associated to C_t^π and $\mathcal{P}_t^\pi$ the projection onto $\mathcal{Q}_t^\pi$, recall (2.2) . The scheme is defined as follows.

Definition 3.1.

(i) The terminal condition $\mathcal{Y}_n^{\Re,\pi}$ is given by a random variable $\xi^\pi \in \mathcal{L}^2(\mathcal{F}_T)$ valued in $\mathcal{Q}_T^\pi$

(ii) for $0 \leqslant i < n$,

$$\begin{cases} \widetilde{\mathcal{Y}}_i^{\Re,\pi} := \mathbb{E}[\mathcal{Y}_{i+1}^{\Re,\pi} \mid \mathcal{F}_{t_i}] + h_i F_i^\pi(\mathcal{Z}_i^{\Re,\pi}), \\ \mathcal{Z}_i^{\Re,\pi} := \mathbb{E}[\mathcal{Y}_{i+1}^{\Re,\pi} H_i \mid \mathcal{F}_{t_i}], \\ \mathcal{Y}_i^{\Re,\pi} := \widetilde{\mathcal{Y}}_i^{\Re,\pi}\mathbf{1}_{\{t_i\notin\Re\}} + \mathcal{P}_{t_i}^\pi(\widetilde{\mathcal{Y}}_i^{\Re,\pi})\mathbf{1}_{\{t_i\in\Re\}}, \end{cases} \tag{3.1}$$

with $(H_i)_{0\leqslant i<n}$ some $\mathbb{R}^{1\times d}$ independent random vectors such that, for all $0 \leqslant i < n$, H_i is $\mathcal{F}_{t_{i+1}}$-measurable, $\mathbb{E}_{t_i}[H_i] = 0$,

$$\lambda_i I_{d\times d} = h_i \mathbb{E}[H_i^\top H_i] = h_i \mathbb{E}_{t_i}[H_i^\top H_i], \tag{3.2}$$

and

$$\frac{\lambda}{d} \leqslant \lambda_i \leqslant \frac{\Lambda}{d}, \tag{3.3}$$

where λ and Λ are positive constants.

Remark 3.1. *Let us remark that* (3.2) *and* (3.3) *imply that*

$$\lambda \leqslant h_i \mathbb{E}[|H_i|^2] = h_i \mathbb{E}_{t_i}[|H_i|^2] \leqslant \Lambda. \tag{3.4}$$

In this section we use following assumptions. (HFd_p)

(i) For all $i \in \{0, ..., n-1\}$, $F_i^\pi : \Omega \times \mathcal{M}^{d,d} \to \mathbb{R}^d$ is a $\mathcal{F}_{t_i} \otimes \mathcal{B}(\mathcal{M}^{d,d})$-measurable function,

(ii) the random cost process C^π satisfies the structure condition (2.1),

(iii) $F_i^{\pi,j}(z) = F_i^{\pi,j}(z^{j\cdot})$ for all $j \in \mathcal{I}$ and all $0 \leqslant i \leqslant n-1$,

(iv) $|F_i^\pi(z) - F_i^\pi(z')| \leqslant L^Z |z - z'|$ for all $z, z' \in \mathcal{M}^{d,d}$,

(v) $\mathbb{E}\left[|\xi^\pi|^2 + \sum_{i=0}^{n-1} |F_i^\pi(0)|^2 h_i + \sup_{t_i \in \Re} |C_{t_i}^\pi|^p\right] \leqslant C_p$,

(vi) $\sup_{0 \leqslant i \leqslant n-1} h_i |H_i| L^Z \leqslant 1$.

Remark 3.2. *i) Under* (HFd_2)*, it is clear that the general scheme* (3.1) *has a unique solution.*

ii) The weights $(H_i)_{0 \leqslant i < n}$ *depend also on the grid* π *but we omit the script* π *for ease of notation.*

We observe that this obliquely reflected backward scheme can be rewritten equivalently for $i \in [\![0, n]\!]$ as

$$\begin{cases} \tilde{\mathcal{Y}}_i^{\Re,\pi} = \xi^\pi + \sum_{k=i}^{n-1} F_k^\pi(\mathcal{Z}_k^{\Re,\pi}) h_k - \sum_{k=i}^{n-1} h_k \lambda_k^{-1} \mathcal{Z}_k^{\Re,\pi} H_k^\top - \sum_{k=i}^{n-1} \Delta\mathcal{M}_k + (\mathcal{K}_n^{\Re,\pi} - \mathcal{K}_i^{\Re,\pi}) \\ \mathcal{K}_k^{\Re,\pi} := \sum_{r=1}^k \Delta\mathcal{K}_r^{\Re,\pi} \text{ with } \Delta\mathcal{K}_r^{\Re,\pi} := \mathcal{Y}_r^{\Re,\pi} - \tilde{\mathcal{Y}}_r^{\Re,\pi}, \end{cases} \tag{3.5}$$

where (λ_k) are given by (3.2) and, for all $k \in [\![0, n-1]\!]$, $\Delta\mathcal{M}_k$ is an $\mathcal{F}_{t_{k+1}}$-measurable random vector satisfying

$$\mathbb{E}_{t_k}[\Delta\mathcal{M}_k] = 0,\ \mathbb{E}_{t_k}[|\Delta\mathcal{M}_k|^2] < \infty \ \text{ and } \ \mathbb{E}_{t_k}[\Delta\mathcal{M}_k H_k] = 0. \tag{3.6}$$

Following Corollary 2.5 in [4], we know that assumption (HFd_p)(v) is an essential ingredient to obtain a comparison result for classical time-discretized BSDE schemes. We are able to adapt this comparison result in the context of obliquely reflected backward scheme in the following proposition.

Proposition 3.1. *Let us consider two obliquely reflected backward schemes solutions* $({}^1\tilde{\mathcal{Y}}^{\Re,\pi}, {}^1\mathcal{Y}^{\Re,\pi}, {}^1\mathcal{Z}^{\Re,\pi})$ *and* $({}^2\tilde{\mathcal{Y}}^{\Re,\pi}, {}^2\mathcal{Y}^{\Re,\pi}, {}^2\mathcal{Z}^{\Re,\pi})$*, associated to generators* $({}^1F_\cdot^\pi)$*,* $({}^2F_\cdot^\pi)$*, terminal conditions* ${}^1\xi^\pi$*,* ${}^2\xi^\pi$ *and random cost processes* $({}^1C^\pi)$*,* $({}^2C^\pi)$ *such that* (HFd_2) *is in force. If*

$${}^1\xi \leqslant {}^2\xi, \qquad {}^1F_i({}^2\mathcal{Z}_i^{\Re,\pi}) \leqslant {}^2F_i({}^2\mathcal{Z}_i^{\Re,\pi}), \quad \textit{for all } \ 0 \leqslant i \leqslant n-1,$$

$$\textit{and} \quad ({}^1C^{\pi}_{t_i})^{jk} \geqslant ({}^2C^{\pi}_{t_i})^{jk}, \quad \textit{for all } \; j,k \in \mathcal{I},\; t_i \in \Re,$$

then we have

$$ {}^1\mathcal{Y}^{\Re,\pi}_i \leqslant {}^2\mathcal{Y}^{\Re,\pi}_i \;\textit{ and }\; {}^1\widetilde{\mathcal{Y}}^{\Re,\pi}_i \leqslant {}^2\widetilde{\mathcal{Y}}^{\Re,\pi}_i, \quad \textit{for all } \; 0 \leqslant i \leqslant n.$$

Moreover, this comparison result stays true if these obliquely reflected backward schemes have two different reflection grids $\Re^1$ and $\Re^2$ with $\Re^1 \subset \Re^2$. In particular, we are allowed to have no projection for the first scheme, i.e. $\Re^1 = \varnothing$.

Proof. We just have to use the comparison theorem for backward schemes (Corollary 2.5 in [4]) and the monotonicity properties of $\mathcal{P}$ (see Remark 2.1). □

Proposition 3.2. *Assume that (HFd_2) is in force. The unique solution $(\widetilde{\mathcal{Y}}^{\Re,\pi}, \mathcal{Y}^{\Re,\pi}, \mathcal{Z}^{\Re,\pi})$ to (3.1) satisfies*

$$\mathbb{E}\left[\sup_{0\leqslant i\leqslant n}\left|\widetilde{\mathcal{Y}}^{\Re,\pi}_i\right|^2 + \sup_{0\leqslant i\leqslant n}\left|\mathcal{Y}^{\Re,\pi}_i\right|^2\right] + \mathbb{E}\left[\sum_{i=0}^{n-1} h_i \left|\mathcal{Z}^{\Re,\pi}_i\right|^2\right] + \mathbb{E}\left[|\mathcal{K}^{\Re,\pi}_n|^2\right] \leqslant C.$$

Proof. The proof of uniform estimates (with respect to n and κ) divides, as usual, in two steps controlling separately $(\widetilde{\mathcal{Y}}^{\Re,\pi}, \mathcal{Y}^{\Re,\pi})$ and $(\mathcal{Z}^{\Re,\pi}, \mathcal{K}^{\Re,\pi})$. It consists in transposing continuous time arguments, see e.g. proof of Theorem 2.4 in [16], in the discrete-time setting.

Step 1. Control of $\widetilde{\mathcal{Y}}^{\Re,\pi}$ and $\mathcal{Y}^{\Re,\pi}$. We consider two non-reflected backward schemes bounding $\widetilde{\mathcal{Y}}^{\Re,\pi}$.

Define the $\mathbb{R}^d$-valued random variable $\breve{\xi}$ and random maps $(\breve{F}_i)_{0\leqslant i\leqslant n-1}$ by $(\breve{\xi})^j := \sum_{k=1}^d \left|(\xi^\pi)^k\right|$ and $(\breve{F}_i)^j(z) := \sum_{k=1}^d \left|(F^\pi_i)^k(z)\right|$ for $1 \leqslant j \leqslant d$ and $0 \leqslant i \leqslant n-1$. We then denote by $(\breve{Y}, \breve{Z})$ the unique solution of the following non-reflected backward scheme:

$$\begin{cases} \breve{Y}_n = \breve{\xi} \\ \breve{Z}_i = \mathbb{E}[\breve{Y}_{i+1} H_i \mid \mathcal{F}_{t_i}], \\ \breve{Y}_i = \mathbb{E}[\breve{Y}_{i+1} \mid \mathcal{F}_{t_i}] + h_i \breve{F}_i(\breve{Z}_i). \end{cases}$$

Since all the components of $\breve{Y}$ are similar, $\breve{Y} \in \mathcal{Q}^\pi$: Thus the above backward scheme is an obliquely reflected backward scheme with same switching costs as in (3.1). We also introduce $(\mathring{Y}, \mathring{Z})$ the solution of the following non-reflected backward scheme

$$\begin{cases} \mathring{Y}_n = \xi^\pi \\ \mathring{Z}_i = \mathbb{E}[\mathring{Y}_{i+1} H_i \mid \mathcal{F}_{t_i}], \\ \mathring{Y}_i = \mathbb{E}[\mathring{Y}_{i+1} \mid \mathcal{F}_{t_i}] + h_i F^\pi_i(\mathring{Z}_i). \end{cases}$$

Using the comparison result given by Proposition 3.1, we straightforwardly deduce that $(\mathring{Y})^j \leqslant (\widetilde{\mathcal{Y}}^{\Re,\pi})^j \leqslant (\mathcal{Y}^{\Re,\pi})^j \leqslant (\breve{Y})^j$, for all $j \in \mathcal{I}$. Since $(\mathring{Y}, \mathring{Z})$ and $(\breve{Y}, \breve{Z})$ are solutions to standard backward schemes, classical estimates and (HFd_2) lead to

$$\begin{aligned}
\mathbb{E}[\sup_{0\leqslant i\leqslant n}|\widetilde{\mathcal{Y}}_i^{\Re,\pi}|^2+\sup_{0\leqslant i\leqslant n}|\mathcal{Y}_i^{\Re,\pi}|^2] &\leqslant \mathbb{E}[\sup_{0\leqslant i\leqslant n}|\mathring{Y}_i|^2+\sup_{0\leqslant i\leqslant n}|\check{Y}_i|^2] \\
&\leqslant C\mathbb{E}\left[|\xi^\pi|^2+\left(\sum_{i=0}^{n-1}|F_i^\pi(0)|^2\,h_i\right)\right] \qquad (3.7)\\
&\leqslant C.
\end{aligned}$$

Step 2. Control of $(\mathcal{Z}^{\Re,\pi},\mathcal{K}^{\Re,\pi})$**.** Let us rewrite (3.5) for $\mathcal{Y}^{\Re,\pi}$ between k and $k+1$ with $k\in[\![0,n-1]\!]$:

$$\mathcal{Y}_k^{\Re,\pi}=\mathcal{Y}_{k+1}^{\Re,\pi}+F_k^\pi(\mathcal{Z}_k^{\Re,\pi})h_k-h_k\lambda_k^{-1}\mathcal{Z}_k^{\Re,\pi}H_k^\top-\Delta\mathcal{M}_k+\Delta\mathcal{K}_k^{\Re,\pi}.$$

Using the identity $|y|^2=|x|^2+2x(y-x)+|x-y|^2$, we obtain, setting $x=\mathcal{Y}_k^{\Re,\pi}$ and $y=\mathcal{Y}_{k+1}^{\Re,\pi}$,

$$\begin{aligned}
|\mathcal{Y}_{k+1}^{\Re,\pi}|^2=&|\mathcal{Y}_k^{\Re,\pi}|^2+2\mathcal{Y}_k^{\Re,\pi}\left(-F_k^\pi(\mathcal{Z}_k^{\Re,\pi})h_k+h_k\lambda_k^{-1}\mathcal{Z}_k^{\Re,\pi}H_k^\top+\Delta\mathcal{M}_k-\Delta\mathcal{K}_k^{\Re,\pi}\right)\\
&+\left|F_k^\pi(\mathcal{Z}_k^{\Re,\pi})h_k-h_k\lambda_k^{-1}\mathcal{Z}_k^{\Re,\pi}H_k^\top-\Delta\mathcal{M}_k+\Delta\mathcal{K}_k^{\Re,\pi}\right|^2.
\end{aligned}$$

Taking the expectation in the previous inequality, we get, combining (HFd_2) with (3.2)-(3.3) and (3.6),

$$\begin{aligned}
\mathbb{E}[|\mathcal{Y}_{k+1}^{\Re,\pi}|^2]\geqslant&\ \mathbb{E}[|\mathcal{Y}_k^{\Re,\pi}|^2]-2\mathbb{E}\left[\mathcal{Y}_k^{\Re,\pi}\left(F_k^\pi(\mathcal{Z}_k^{\Re,\pi})h_k+\Delta\mathcal{K}_k^{\Re,\pi}\right)\right]\\
&+\mathbb{E}\left[\left|h_k\lambda_k^{-1}\mathcal{Z}_k^{\Re,\pi}H_k^\top\right|^2\right]+\mathbb{E}[|\Delta\mathcal{M}_k|^2]\\
\geqslant&\ \mathbb{E}[|\mathcal{Y}_k^{\Re,\pi}|^2]-C\mathbb{E}\left[|\mathcal{Y}_k^{\Re,\pi}|\left(|F_k^\pi(0)|h_k+|\mathcal{Z}_k^{\Re,\pi}|h_k\right)\right]-2\mathbb{E}\left[\mathcal{Y}_k^{\Re,\pi}\Delta\mathcal{K}_k^{\Re,\pi}\right]\\
&+\mathbb{E}\left[h_k^2\lambda_k^{-2}\mathbb{E}_{t_k}\left[\sum_{i,j\in[\![1,d]\!]}((\mathcal{Z}_k^{\Re,\pi})^\top\mathcal{Z}_k^{\Re,\pi})^{ij}(H_k)^{1i}(H_k)^{1j}\right]\right]+\mathbb{E}[|\Delta\mathcal{M}_k|^2]\\
\geqslant&\ \mathbb{E}[|\mathcal{Y}_k^{\Re,\pi}|^2]-C\mathbb{E}\left[|\mathcal{Y}_k^{\Re,\pi}|\left(|F_k^\pi(0)|h_k+|\mathcal{Z}_k^{\Re,\pi}|h_k\right)\right]-2\mathbb{E}\left[\mathcal{Y}_k^{\Re,\pi}\Delta\mathcal{K}_k^{\Re,\pi}\right]\\
&+\frac{d}{\Lambda}\mathbb{E}\left[h_k|\mathcal{Z}_k^{\Re,\pi}|^2\right]+\mathbb{E}[|\Delta\mathcal{M}_k|^2].
\end{aligned}$$

Then we sum over $k \in [\![0, n-1]\!]$ and we compute, using Young inequality with $\varepsilon > 0$,

$$\begin{aligned}\sum_{k=0}^{n-1} \mathbb{E}\left[h_k|\mathcal{Z}_k^{\Re,\pi}|^2\right] + \sum_{k=0}^{n-1} \mathbb{E}[|\Delta\mathcal{M}_k|^2] &\leqslant C_\varepsilon \mathbb{E}\left[\sup_{0\leqslant k\leqslant n} |\mathcal{Y}_k^{\Re,\pi}|^2 + \sum_{k=0}^{n-1} |F_k^\pi(0)|^2 h_k\right] \\ &\quad + \varepsilon \sum_{k=0}^{n-1} \mathbb{E}\left[h_k|\mathcal{Z}_k^{\Re,\pi}|^2\right] + 2\mathbb{E}\left[\sup_{0\leqslant k\leqslant n} |\mathcal{Y}_k^{\Re,\pi}||\mathcal{K}_n^{\Re,\pi}|\right] \\ &\leqslant C_\varepsilon \mathbb{E}\left[\sup_{0\leqslant k\leqslant n} |\mathcal{Y}_k^{\Re,\pi}|^2 + \sum_{k=0}^{n-1} |F_k^\pi(0)|^2 h_k\right] \\ &\quad + \varepsilon \sum_{k=0}^{n-1} \mathbb{E}\left[h_k|\mathcal{Z}_k^{\Re,\pi}|^2\right] + \varepsilon\mathbb{E}\left[|\mathcal{K}_n^{\Re,\pi}|^2\right]. \end{aligned} \tag{3.8}$$

Moreover, we get from (3.5)

$$\begin{aligned}\mathbb{E}\left[|\mathcal{K}_n^{\Re,\pi}|^2\right] &\leqslant C\mathbb{E}\left[\sup_{0\leqslant k\leqslant n} |\widetilde{\mathcal{Y}}_k^{\Re,\pi}|^2 + \sum_{k=0}^{n-1} |F_k^\pi(0)|^2 h_k\right] \\ &\quad + C\sum_{k=0}^{n-1} \mathbb{E}\left[h_k|\mathcal{Z}_k^{\Re,\pi}|^2\right] + C\sum_{k=0}^{n-1} \mathbb{E}[|\Delta\mathcal{M}_k|^2]. \end{aligned} \tag{3.9}$$

Combining (3.8) with (3.9), and using (HFd_2) and (3.7), classical calculations yield, for ε small enough,

$$\sum_{k=0}^{n-1} \mathbb{E}\left[h_k|\mathcal{Z}_k^{\Re,\pi}|^2\right] + \sum_{k=0}^{n-1} \mathbb{E}[|\Delta\mathcal{M}_k|^2] \leqslant C.$$

Finally we can insert this last inequality into (3.9) and use once again (HFd_2) and (3.7) to conclude the proof. □

3.2 Optimal switching problem representation

We now introduce a discrete-time version of the switching problem, which will allow us to give a new representation of the scheme given in Definition 3.1. To simplify notations, we start by adapting the definition of switching strategies to the discrete-time setting: A switching strategy a is now a nondecreasing sequence of stopping times $(\theta_r)_{r\in\mathbb{N}}$ valued in $\mathbb{N}$, combined with a sequence of random variables $(\alpha_r)_{r\in\mathbb{N}}$ valued in $\mathcal{I}$, such that α_r is $\mathcal{F}_{t_{\theta_r}}$-measurable for any $r \in \mathbb{N}$.

Then by mimicking Section 2.1, we define classical objects related to switching strategies. For a switching strategy $a = (\theta_r, \alpha_r)_{r\in\mathbb{N}}$, we introduce $\mathcal{N}^a$ the (random) number of switches before n:

$$\mathcal{N}^a = \#\{r \in \mathbb{N}^* \ : \ \theta_r \leqslant n\}\ . \tag{3.10}$$

To any switching strategy $a = (\theta_r, \alpha_r)_{r\in\mathbb{N}}$, we associate the current state process $(a_i)_{i\in[\![0,n]\!]}$ and the cumulative cost process $(\mathcal{A}^a_i)_{i\in[\![0,n]\!]}$ defined respectively by

$$a_i \ := \ \alpha_0 1_{\{0\leqslant i<\theta_0\}} + \sum_{r=1}^{\mathcal{N}^a} \alpha_{r-1} 1_{\{\theta_{r-1}\leqslant i<\theta_r\}} \ \text{ and } \ \mathcal{A}^a_i \ := \ \sum_{r=1}^{\mathcal{N}^a} (C^\pi_{t_{\theta_r}})^{\alpha_{r-1}\alpha_r} 1_{\{\theta_r\leqslant i\leqslant n\}} ,$$

for $0 \leqslant i \leqslant n$. We denote by $\mathscr{A}^{\mathfrak{R},\pi}$ the set of $\mathfrak{R}$-admissible strategies:

$$\mathscr{A}^{\mathfrak{R},\pi} = \{a = (\theta_r, \alpha_r)_{r\in\mathbb{N}} \text{ switching strategy } \mid t_{\theta_r} \in \mathfrak{R} \quad \forall r \in [\![1, \mathcal{N}^a]\!], \ \mathbb{E}\big[|\mathcal{A}^a_n|^2\big] < \infty\} \ .$$

For $(i, j) \in [\![0, n]\!] \times \mathcal{I}$, the set $\mathscr{A}^{\mathfrak{R},\pi}_{i,j}$ of admissible strategies starting from j at time t_i is defined by

$$\mathscr{A}^{\mathfrak{R},\pi}_{i,j} = \{a = (\theta_r, \alpha_r)_{r\in\mathbb{N}} \in \mathscr{A}^{\mathfrak{R},\pi} \ |\theta_0 = i, \ \alpha_0 = j\} \ .$$

For a strategy $a \in \mathscr{A}^{\mathfrak{R},\pi}_{i,j}$ we define the one dimensional $\mathfrak{R}$-switched backward scheme whose solution $(\mathcal{U}^{\mathfrak{R},\pi,a}, \mathcal{V}^{\mathfrak{R},\pi,a})$ satisfies

$$\begin{cases} \mathcal{U}^{\mathfrak{R},\pi,a}_n &= \xi^{\pi,a_n} \\ \mathcal{V}^{\mathfrak{R},\pi,a}_k &= \mathbb{E}[\mathcal{U}^{\mathfrak{R},\pi,a}_{k+1} H^\top_k \mid \mathcal{F}_{t_k}], \\ \mathcal{U}^{\mathfrak{R},\pi,a}_k &= \mathbb{E}[\mathcal{U}^{\mathfrak{R},\pi,a}_{k+1} \mid \mathcal{F}_{t_k}] + h_k F^{\pi,a_k}_k (\mathcal{V}^{\mathfrak{R},\pi,a}_k) - \sum_{j=1}^{\mathcal{N}^a} (C^\pi_{t_{\theta_j}})^{\alpha_{j-1}\alpha_j} 1_{\theta_j \leqslant k}, \quad i \leqslant k < n. \end{cases} \tag{3.11}$$

Similarly to equation (3.1), we observe that this obliquely reflected backward scheme can be rewritten equivalently for $k \in [\![i, n]\!]$ as

$$\begin{aligned} \mathcal{U}^{\mathfrak{R},\pi,a}_k =& \xi^{\pi,a_n} + \sum_{m=k}^{n-1} F^{\pi,a_m}_m (\mathcal{V}^{\mathfrak{R},\pi,a}_m) h_m - \sum_{m=k}^{n-1} h_m \lambda^{-1}_m \mathcal{V}^{\mathfrak{R},\pi,a}_m H^\top_m - \sum_{m=k}^{n-1} \Delta\mathcal{M}^a_m \\ &- \mathcal{A}^a_n + \mathcal{A}^a_k \end{aligned} \tag{3.12}$$

where (λ_k) are given by (3.2) and, for all $k \in [\![0, n-1]\!]$, $\Delta\mathcal{M}^a_k$ is an $\mathcal{F}_{t_{k+1}}$-measurable random variable satisfying

$$\mathbb{E}_{t_k}[\Delta\mathcal{M}^a_k] = 0, \ \mathbb{E}_{t_k}[|\Delta\mathcal{M}^a_k|^2] < \infty \ \text{ and } \mathbb{E}_{t_k}[\Delta\mathcal{M}^a_k H_k] = 0. \tag{3.13}$$

The next theorem is a Snell envelope representation of the obliquely reflected backward scheme.

Proposition 3.3. *For any $j \in \mathcal{I}$ and $0 \leqslant i \leqslant n$, the following hold:*

(i) The discrete process $\mathcal{Y}^{\mathfrak{R},\pi}$ dominates any $\mathfrak{R}$-switched backward scheme, that is,

$$\mathcal{U}^{\mathfrak{R},\pi,a}_i \leqslant (\widetilde{\mathcal{Y}}^{\mathfrak{R},\pi}_i)^j, \quad \mathbb{P}\text{-a.s. for any } a \in \mathscr{A}^{\mathfrak{R},\pi}_{i,j}. \tag{3.14}$$

(ii) Define the strategy $\bar{a}^{\Re,\pi} = (\bar{\theta}_r, \bar{\alpha}_r)_{r\geqslant 0}$ *recursively by* $(\bar{\theta}_0, \bar{\alpha}_0) := (i,j)$ *and, for* $r \geqslant 1$,

$$\bar{\theta}_r := \inf\left\{k \in [\![\bar{\theta}_{r-1}, n]\!] \;\middle|\; t_k \in \Re,\ (\widetilde{\mathcal{Y}}_k^{\Re,\pi})^{\bar{\alpha}_{r-1}} \leqslant \max_{m\neq\bar{\alpha}_{r-1}} \{(\widetilde{\mathcal{Y}}_k^{\Re,\pi})^m - C_{t_k}^{\bar{\alpha}_{r-1}m}\}\right\},$$

$$\bar{\alpha}_r := \min\left\{\ell \neq \bar{\alpha}_{r-1} \;\middle|\; (\widetilde{\mathcal{Y}}_{\theta_r}^{\Re,\pi})^\ell - C_{t_{\theta_r}}^{\bar{\alpha}_{r-1}\ell} = \max_{m\neq\bar{\alpha}_{r-1}} \{(\widetilde{\mathcal{Y}}_{\theta_r}^{\Re,\pi})^m - C_{t_{\theta_r}}^{\bar{\alpha}_{r-1}m}\}\right\}$$

Then we have $\bar{a}^{\Re,\pi} \in \mathscr{A}_{i,j}^{\Re,\pi}$ *and*

$$(\widetilde{\mathcal{Y}}_i^{\Re,\pi})^j = \mathcal{U}_i^{\Re,\pi,\bar{a}^{\Re,\pi}} \quad \mathbb{P}\text{-}a.s. \tag{3.15}$$

(iii) The following "Snell envelope" representation holds:

$$(\widetilde{\mathcal{Y}}_i^{\Re,\pi})^j = \operatorname*{ess\,sup}_{a\in\mathscr{A}_{i,j}^{\Re,\pi}} \mathcal{U}_i^{\Re,\pi,a} \quad \mathbb{P}\text{-}a.s. \tag{3.16}$$

Proof. We will adapt the proof of Theorem 2.1 in [8] to the discrete time setting. Observe first that assertion (iii) is a direct consequence of (i) and (ii).

Let us fix $i \in [\![0,n]\!]$ and $j \in \mathcal{I}$.

Step 1. We first prove (i).

Set $a = (\theta_r, \alpha_r)_{r\geqslant 0} \in \mathscr{A}_{i,j}^{\Re,\pi}$ and the process $(\widetilde{\mathcal{Y}}^a, \mathcal{Z}^a)$ defined, for $k \in [\![i,n]\!]$, by

$$\begin{cases} \widetilde{\mathcal{Y}}_k^a & := \sum_{r\geqslant 0} (\widetilde{\mathcal{Y}}_k^{\Re,\pi})^{\alpha_r} 1_{\theta_r \leqslant k < \theta_{r+1}} + \xi^{\pi,a_n} 1_{k=n} \\ \mathcal{Z}_k^a & := \sum_{r\geqslant 0} (\mathcal{Z}_k^{\Re,\pi})^{\alpha_r} 1_{\theta_r \leqslant k < \theta_{r+1}}. \end{cases} \tag{3.17}$$

Observe that these processes jump between the components of the obliquely reflected backward scheme (3.5) according to the strategy a, and, between two jumps, we have

$$\begin{aligned} \widetilde{\mathcal{Y}}_{\theta_r}^a &= (\mathcal{Y}_{\theta_{r+1}}^{\Re,\pi})^{\alpha_r} + \sum_{k=\theta_r}^{\theta_{r+1}-1} F_k^{\pi,\alpha_r}((\mathcal{Z}_k^{\Re,\pi})^{\alpha_r}) h_k - \sum_{k=\theta_r}^{\theta_{r+1}-1} h_k \lambda_k^{-1} (\mathcal{Z}_k^{\Re,\pi})^{\alpha_r} H_k^\top \\ &\quad - \sum_{k=\theta_r}^{\theta_{r+1}-1} \Delta(\mathcal{M}_k)^{\alpha_r} + (\mathcal{K}_{(\theta_{r+1}-1)\vee\theta_r}^{\Re,\pi})^{\alpha_r} - (\mathcal{K}_{\theta_r}^{\Re,\pi})^{\alpha_r} \\ &= \widetilde{\mathcal{Y}}_{\theta_{r+1}}^a + \sum_{k=\theta_r}^{\theta_{r+1}-1} F_k^{\pi,a_k}(\mathcal{Z}_k^a) h_k - \sum_{k=\theta_r}^{\theta_{r+1}-1} h_k \lambda_k^{-1} \mathcal{Z}_k^a H_k^\top - \sum_{k=\theta_r}^{\theta_{r+1}-1} \Delta(\mathcal{M}_k)^{\alpha_r} \\ &\quad + (\mathcal{K}_{(\theta_{r+1}-1)\vee\theta_r}^{\Re,\pi})^{\alpha_r} - (\mathcal{K}_{\theta_r}^{\Re,\pi})^{\alpha_r} + ((\mathcal{Y}_{\theta_{r+1}}^{\Re,\pi})^{\alpha_r} - (\widetilde{\mathcal{Y}}_{\theta_{r+1}}^{\Re,\pi})^{\alpha_{r+1}}), \qquad r \geqslant 0. \end{aligned} \tag{3.18}$$

Introducing

$$\begin{aligned} \mathcal{K}_k^a := \sum_{r=0}^{\mathcal{N}^a-1} \Bigg[& \sum_{m=\theta_r\wedge k}^{(\theta_{r+1}-1)\vee\theta_r\wedge k} (\Delta\mathcal{K}_m^{\Re,\pi})^{\alpha_r} \\ & + 1_{\theta_{r+1}\leqslant k} \left((\mathcal{Y}_{\theta_{r+1}}^{\Re,\pi})^{\alpha_r} - (\widetilde{\mathcal{Y}}_{\theta_{r+1}}^{\Re,\pi})^{\alpha_{r+1}} + (C_{t_{\theta_{r+1}}}^\pi)^{\alpha_r\alpha_{r+1}} \right) \Bigg] \end{aligned}$$

for $k \in [\![i,n]\!]$, and summing up (3.18) over r, we get, for $k \in [\![i,n]\!]$,

$$\begin{aligned}\widetilde{\mathcal{Y}}^a_k =& \xi^{\pi,a_n} + \sum_{m=k}^{n-1} F^{\pi,a_m}_m(\mathcal{Z}^a_m)h_m - \sum_{m=k}^{n-1} h_m\lambda_m^{-1}\mathcal{Z}^a_m H_m^\top - \sum_{r=0}^{\mathcal{N}^a-1}\sum_{m=\theta_r\vee k}^{(\theta_{r+1}-1)\vee k}\Delta(\mathcal{M}_m)^{\alpha_r}\\ &- \mathcal{A}^a_n + \mathcal{A}^a_k + \mathcal{K}^a_n - \mathcal{K}^a_k.\end{aligned}$$

Using the relation $\mathcal{Y}^{\Re,\pi}_{\theta_r} = \mathcal{P}_{\theta_k}(\widetilde{\mathcal{Y}}^{\Re,\pi}_{\theta_k})$ for all $r \in [\![0,\mathcal{N}^a]\!]$, we easily check that $\mathcal{K}^a$ is an increasing process. Since $\mathcal{U}^{\Re,\pi,a}$ solves (3.12), we deduce by a comparison argument (see Corollary 2.5 in [4]) that $\mathcal{U}^{\Re,\pi,a}_i \leqslant \widetilde{\mathcal{Y}}^a_i$. Since a is arbitrary in $\mathscr{A}^{\Re,\pi}_{i,j}$, we deduce (3.14).

Step 2. We now prove (ii).

Consider the strategy $\bar{a}^{\Re,\pi}$ given above as well as the associated process $(\widetilde{\mathcal{Y}}^{\bar{a}^{\Re,\pi}}, \mathcal{Z}^{\bar{a}^{\Re,\pi}})$ defined as in (3.17). By definition of $\bar{a}^{\Re,\pi}$, we have

$$(\mathcal{Y}^{\Re,\pi}_{\bar{\theta}_{r+1}})^{\bar{\alpha}_r} = (\mathcal{P}^\pi_{\bar{\theta}_{r+1}}(\widetilde{\mathcal{Y}}^{\Re,\pi}_{\bar{\theta}_{r+1}}))^{\bar{\alpha}_r} = (\widetilde{\mathcal{Y}}^{\Re,\pi}_{\bar{\theta}_{r+1}})^{\bar{\alpha}_{r+1}} - (C^\pi_{t_{\bar{\alpha}_{r+1}}})^{\bar{\alpha}_r\bar{\alpha}_{r+1}}, \quad r \geqslant 0,$$

which gives that $\mathcal{K}^{\bar{a}^{\Re,\pi}} = 0$ and then, for all $k \in [\![i,n]\!]$,

$$\begin{aligned}\widetilde{\mathcal{Y}}^{\bar{a}^{\Re,\pi}}_k =& \xi^{\pi,\bar{a}^{\Re,\pi}_n} + \sum_{m=k}^{n-1} F^{\pi,\bar{a}^{\Re,\pi}_m}_m(\mathcal{Z}^{\bar{a}^{\Re,\pi}}_m)h_m - \sum_{m=k}^{n-1} h_m\lambda_m^{-1}\mathcal{Z}^{\bar{a}^{\Re,\pi}}_m H_m^\top\\ &- \sum_{r=0}^{\mathcal{N}^{\bar{a}^{\Re,\pi}}-1}\sum_{m=\bar{\theta}_r\vee k}^{(\bar{\theta}_{r+1}-1)\vee k}(\Delta\mathcal{M}_m)^{\bar{\alpha}_r} - \mathcal{A}^{\bar{a}^{\Re,\pi}}_n + \mathcal{A}^{\bar{a}^{\Re,\pi}}_k.\end{aligned} \tag{3.19}$$

Hence, $(\widetilde{\mathcal{Y}}^{\bar{a}^{\Re,\pi}}, \mathcal{Z}^{\bar{a}^{\Re,\pi}})$ and $(\mathcal{U}^{\bar{a}^{\Re,\pi}}, \mathcal{V}^{\bar{a}^{\Re,\pi}})$ are solutions of the same backward scheme and $(\widetilde{\mathcal{Y}}^{\Re,\pi}_i)^j = \mathcal{U}^{\bar{a}^{\Re,\pi}}_i$. To complete the proof, we only need to check that $\bar{a}^{\Re,\pi} \in \mathscr{A}^{\Re,\pi}$, that is $\mathbb{E}[|\mathcal{A}^{\bar{a}^{\Re,\pi}}_n|^2] < \infty$. By definition of $\bar{a}^{\Re,\pi}$ on $[\![i,n]\!]$ and the structure condition on costs (2.1), we have $|\mathcal{A}^{\bar{a}^{\Re,\pi}}_i| \leqslant \max_{k\neq j}|C^{jk}_{t_i}|$ which gives $\mathbb{E}[|\mathcal{A}^{\bar{a}^{\Re,\pi}}_i|^2] \leqslant C$. Combining (3.19) with the Lipschitz property of F^π and estimates in Proposition 3.2, we get the square integrability of $\mathcal{A}^{\bar{a}^{\Re,\pi}}_n$ and the proof is complete. $\square$

Proposition 3.4. *Assume than* (HFd_2) *is in force. For all* $0 \leqslant i \leqslant n$, $j \in \mathcal{I}$, *we have*

$$\mathbb{E}\left[\sup_{i\leqslant k\leqslant n}\left|\mathcal{U}^{\Re,\pi,\bar{a}^{\Re,\pi}}_k\right|^2 + \sum_{k=i}^{n-1} h_k\left|\mathcal{V}^{\Re,\pi,\bar{a}^{\Re,\pi}}_k\right|^2 + \left|\mathcal{A}^{\bar{a}^{\Re,\pi}}_n\right|^2 + \left|\mathcal{N}^{\bar{a}^{\Re,\pi}}\right|^2\right] \leqslant C,$$

for the optimal strategy $\bar{a}^{\Re,\pi} \in \mathscr{A}^{\Re}_{i,j}$.

Proof. Fix $(i,j) \in [\![0,n]\!] \times \mathcal{I}$. According to the identification of $(\mathcal{U}^{\Re,\pi,\bar{a}^{\Re,\pi}}, \mathcal{V}^{\Re,\pi,\bar{a}^{\Re,\pi}})$ with $(\widetilde{\mathcal{Y}}^{\bar{a}^{\Re,\pi}}, \mathcal{Z}^{\bar{a}^{\Re,\pi}})$ obtained in the proof of Proposition 3.3, we deduce from Proposition 3.2 expected controls on $\mathcal{U}^{\Re,\pi,\bar{a}^{\Re,\pi}}$ and $\mathcal{V}^{\Re,\pi,\bar{a}^{\Re,\pi}}$.

By taking conditional expectation in (3.19), we have

$$\mathbb{E}_{t_i}[\mathcal{A}_n^{\bar{a}^{\Re,\pi}}] = \mathbb{E}_{t_i}\left[\xi^{\pi,\bar{a}_n^{\Re,\pi}} - \widetilde{\mathcal{Y}}_i^{\bar{a}^{\Re,\pi}} + \sum_{m=i}^{n-1} F_m^{\pi,\bar{a}_m^{\Re,\pi}}(\mathcal{Z}_m^{\bar{a}^{\Re,\pi}})h_m + \mathcal{A}_i^{\bar{a}^{\Re,\pi}}\right].$$

Thus, using standard inequalities and the growth of F^π, we easily obtain

$$\mathbb{E}[|\mathcal{A}_n^{\bar{a}^{\Re,\pi}}|^2] \leqslant C\mathbb{E}\left[\sup_{i\leqslant k\leqslant n}|\widetilde{\mathcal{Y}}_k^{\bar{a}^{\Re,\pi}}|^2 + \sum_{m=i}^{n-1}|\mathcal{Z}_m^{\bar{a}^{\Re,\pi}}|^2 h_m + |\mathcal{A}_i^{\bar{a}^{\Re,\pi}}|^2\right].$$

We have already noticed in the proof of Proposition 3.3 that we have $|\mathcal{A}_i^{\bar{a}^{\Re,\pi}}| \leqslant \max_{k\neq j}|C_i^{jk}|$, which inserted into the previous inequality leads to $\mathbb{E}[|\mathcal{A}_n^{\bar{a}^{\Re,\pi}}|^2] \leqslant C$.

We finally complete the proof, observing from the structure condition (2.1) that

$$\mathbb{E}[|\mathcal{N}^{\bar{a}^{\Re,\pi}}|^2] \leqslant C\mathbb{E}[|\mathcal{A}_n^{\bar{a}^{\Re,\pi}}|^2].$$

□

3.3 Stability of obliquely reflected backward schemes

We now consider two obliquely reflected backward schemes, with different parameters but the same reflection grid $\Re$. For $\ell \in \{1,2\}$, we consider an $\mathcal{F}_T$-measurable random terminal condition ${}^\ell\xi$, a random generator $z \mapsto {}^\ell F(.,z)$ and random cost processes $({}^\ell C^{ij})_{1\leqslant i,j\leqslant d}$ satisfying the structural condition (2.1). As in Subsection 3.2, terminal conditions, generators and cost processes are allowed to depend on π but we omit the script π for reading convenience. We denote by $({}^\ell\widetilde{\mathcal{Y}}^{\Re,\pi}, {}^\ell\mathcal{Y}^{\Re,\pi}, {}^\ell\mathcal{Z}^{\Re,\pi})$ the solution of the associated obliquely reflected backward scheme.

Defining $\delta\mathcal{Y}^{\Re,\pi} := {}^1\mathcal{Y}^{\Re,\pi} - {}^2\mathcal{Y}^{\Re,\pi}$, $\delta\widetilde{\mathcal{Y}}^{\Re,\pi} := {}^1\widetilde{\mathcal{Y}}^{\Re,\pi} - {}^2\widetilde{\mathcal{Y}}^{\Re,\pi}$, $\delta\mathcal{Z}^{\Re,\pi} := {}^1\mathcal{Z}^{\Re,\pi} - {}^2\mathcal{Z}^{\Re,\pi}$, $\delta\xi := {}^1\xi - {}^2\xi$ together with

$$\begin{aligned}
|\delta C_t|_\infty &:= \max_{i,j\in\mathcal{I}}\left|{}^1C_t^{ij} - {}^2C_t^{ij}\right|,\\
|\delta F_k|_\infty &:= \max_{i\in\mathcal{I}}\sup_{z\in\mathcal{M}^{d,d}}\left|{}^1F_k^i - {}^2F_k^i\right|(z),
\end{aligned}$$

for $0 \leqslant k \leqslant n-1$, we prove the following stability result.

Proposition 3.5. *Assume that (HFd_p) is in force for some given $p \geqslant 2$. Then we have, for any $i \in [\![0,n]\!]$,*

$$\begin{aligned}
&\sup_{i\leqslant k\leqslant n}\mathbb{E}\left[\left|\delta\mathcal{Y}_k^{\Re,\pi}\right|^2 + \left|\delta\widetilde{\mathcal{Y}}_k^{\Re,\pi}\right|^2\right] + \frac{1}{\kappa}\mathbb{E}\left[\sum_{k=i}^{n-1}h_k\left|\delta\mathcal{Z}_k^{\Re,\pi}\right|^2\right]\\
&\leqslant C\mathbb{E}\left[\sum_{k=i}^{n-1}|\delta F_k|_\infty^2 h_k + |\delta\xi|^2\right] + C_p\kappa^{4/p}\mathbb{E}\left[\sup_{0\leqslant k\leqslant n, t_k\in\Re}|\delta C_{t_k}|_\infty^p\right]^{2/p}.
\end{aligned}$$

Proof. We adapt to our setting the proof of Proposition 2.3 in [8]. The proof is divided into three steps and relies heavily on the reinterpretation in terms of switching problems. We first introduce a convenient dominating process and then provide successively the controls on $\delta\mathcal{Y}^{\Re,\pi}$ and $\delta\mathcal{Z}^{\Re,\pi}$ terms.

Step 1. Introduction of an auxiliary backward scheme. Let us define $F := {}^1F \vee {}^2F$, $\xi := {}^1\xi \vee {}^2\xi$ and C by $C^{ij} := {}^1C^{ij} \vee {}^2C^{ij}$. Observe (HFd_p) holds for the data (C, F, ξ) and C satisfies the structure condition (2.1). We denote by $(\widetilde{\mathcal{Y}}^{\Re,\pi}, \mathcal{Y}^{\Re,\pi}, \mathcal{Z}^{\Re,\pi})$ the solution of the discretely obliquely reflected backward scheme with generator F, terminal condition ξ, reflection grid $\Re$ and cost process C.

Using Proposition 3.1 and the definition of F, ξ and C, we obtain that

$$\widetilde{\mathcal{Y}}^{\Re,\pi} \geqslant {}^1\widetilde{\mathcal{Y}}^{\Re,\pi} \vee {}^2\widetilde{\mathcal{Y}}^{\Re,\pi}. \tag{3.20}$$

Using Proposition 3.3, we introduce switched backward schemes associated to ${}^1\mathcal{Y}^{\Re,\pi}$, ${}^2\mathcal{Y}^{\Re,\pi}$ and $\mathcal{Y}^{\Re,\pi}$ and denote by $\check{a} = (\check{\theta}_r, \check{\alpha}_r)_{r\geqslant 0}$ the optimal strategy related to $\mathcal{Y}^{\Re,\pi}$ starting from a fixed $(i,j) \in [\![0,n]\!] \times \mathcal{I}$. therefore, we have

$$\begin{aligned}(\widetilde{\mathcal{Y}}_i^{\Re,\pi})^j = \mathcal{U}_i^{\Re,\pi,\check{a}} =& \xi^{\check{a}_n} + \sum_{k=i}^{n-1} F_k^{\check{a}_k}(\mathcal{V}_k^{\Re,\pi,\check{a}})h_k - \sum_{k=i}^{n-1} h_k\lambda_k^{-1}\mathcal{V}_k^{\Re,\pi,\check{a}}H_k^\top - \sum_{k=i}^{n-1}\Delta\mathcal{M}_k^{\check{a}} \\ &- \mathcal{A}_n^{\check{a}} + \mathcal{A}_i^{\check{a}}\end{aligned} \tag{3.21}$$

Step 2. Stability of the Y component. Since $\check{a} \in \mathscr{A}_{i,j}^{\Re,\pi}$, we deduce from Proposition 3.3 (i) that

$$\begin{aligned}({}^\ell\widetilde{\mathcal{Y}}_i^{\Re,\pi})^j \geqslant {}^\ell\mathcal{U}_i^{\Re,\pi,\check{a}} =& {}^\ell\xi^{\check{a}_n} + \sum_{k=i}^{n-1} {}^\ell F_k^{\check{a}_k}({}^\ell\mathcal{V}_k^{\Re,\pi,\check{a}})h_k - \sum_{k=i}^{n-1} h_k\lambda_k^{-1}{}^\ell\mathcal{V}_k^{\Re,\pi,\check{a}}H_k^\top - \sum_{k=i}^{n-1}\Delta{}^\ell\mathcal{M}_k^{\check{a}} \\ &- {}^\ell\mathcal{A}_n^{\check{a}} + {}^\ell\mathcal{A}_i^{\check{a}}, \qquad \ell \in \{1,2\},\end{aligned} \tag{3.22}$$

where ${}^\ell\mathcal{A}^{\check{a}}$ is the process of cumulated costs $({}^\ell C^{ij})_{i,j\in\mathcal{I}}$ associated to the strategy $\check{a}$. Combining this estimate with (3.20) and (3.21), we derive

$$|({}^1\widetilde{\mathcal{Y}}_i^{\Re,\pi})^j - ({}^2\widetilde{\mathcal{Y}}_i^{\Re,\pi})^j| \leqslant |\mathcal{U}_i^{\Re,\pi,\check{a}} - {}^1\mathcal{U}_i^{\Re,\pi,\check{a}}| + |\mathcal{U}_i^{\Re,\pi,\check{a}} - {}^2\mathcal{U}_i^{\Re,\pi,\check{a}}|. \tag{3.23}$$

Since both terms on the right-hand side of (3.23) are treated similarly, we focus on the first one and introduce discrete processes $\Gamma^{\check{a}} := \mathcal{U}^{\Re,\pi,\check{a}} + \mathcal{A}^{\check{a}}$ and ${}^1\Gamma^{\check{a}} := {}^1\mathcal{U}^{\Re,\pi,\check{a}} + {}^1\mathcal{A}^{\check{a}}$. Rewriting (3.21) and (3.22) between k and $k+1$ for $k \in [\![i, n-1]\!]$, we get

$$\begin{aligned}\Gamma_k^{\check{a}} - {}^1\Gamma_k^{\check{a}} = &\Gamma_{k+1}^{\check{a}} - {}^1\Gamma_{k+1}^{\check{a}} + [F_k^{\check{a}_k}(\mathcal{V}_k^{\Re,\pi,\check{a}}) - {}^1F_k^{\check{a}_k}({}^1\mathcal{V}_k^{\Re,\pi,\check{a}})]h_k \\ &- h_k\lambda_k^{-1}[\mathcal{V}_k^{\Re,\pi,\check{a}} - {}^1\mathcal{V}_k^{\Re,\pi,\check{a}}]H_k^\top - [\Delta\mathcal{M}_k^{\check{a}} - \Delta^1\mathcal{M}_k^{\check{a}}].\end{aligned}$$

Using the identity $|y|^2 = |x|^2 + 2x(y-x) + |x-y|^2$, we obtain,

$$\begin{aligned}&\mathbb{E}_{t_k}[|\Gamma_{k+1}^{\check{a}} - {}^1\Gamma_{k+1}^{\check{a}}|^2] \\ &= |\Gamma_k^{\check{a}} - {}^1\Gamma_k^{\check{a}}|^2 - 2(\Gamma_k^{\check{a}} - {}^1\Gamma_k^{\check{a}})(F_k^{\check{a}_k}(\mathcal{V}_k^{\Re,\pi,\check{a}}) - {}^1F_k^{\check{a}_k}({}^1\mathcal{V}_k^{\Re,\pi,\check{a}}))h_k \\ &\quad + \mathbb{E}_{t_k}[|[F_k^{\check{a}_k}(\mathcal{V}_k^{\Re,\pi,\check{a}}) - {}^1F_k^{\check{a}_k}({}^1\mathcal{V}_k^{\Re,\pi,\check{a}})]h_k - h_k\lambda_k^{-1}[\mathcal{V}_k^{\Re,\pi,\check{a}} - {}^1\mathcal{V}_k^{\Re,\pi,\check{a}}]H_k^\top - [\Delta\mathcal{M}_k^{\check{a}} - \Delta^1\mathcal{M}_k^{\check{a}}]|^2].\end{aligned}$$

Then, by the same reasoning as in the step 2 of the proof of Proposition 3.2, previous equality becomes

$$\mathbb{E}_{t_k}[|\Gamma^{\check{a}}_{k+1} - {}^1\Gamma^{\check{a}}_{k+1}|^2] \geqslant |\Gamma^{\check{a}}_k - {}^1\Gamma^{\check{a}}_k|^2 - 2(\Gamma^{\check{a}}_k - {}^1\Gamma^{\check{a}}_k)(F^{\check{a}_k}_k(\mathcal{V}^{\Re,\pi,\check{a}}_k) - {}^1F^{\check{a}_k}_k({}^1\mathcal{V}^{\Re,\pi,\check{a}}_k))h_k + \frac{d}{\Lambda}h_k|\mathcal{V}^{\Re,\pi,\check{a}}_k - {}^1\mathcal{V}^{\Re,\pi,\check{a}}_k|^2,$$

and we obtain, by summing over k and taking expectation,

$$\mathbb{E}\left[|\Gamma^{\check{a}}_i - {}^1\Gamma^{\check{a}}_i|^2 + \sum_{k=i}^{n-1} h_k|\mathcal{V}^{\Re,\pi,\check{a}}_k - {}^1\mathcal{V}^{\Re,\pi,\check{a}}_k|^2\right] \leqslant C\mathbb{E}\left[|\Gamma^{\check{a}}_n - {}^1\Gamma^{\check{a}}_n|^2 + \sum_{k=i}^{n-1}|\Gamma^{\check{a}}_k - {}^1\Gamma^{\check{a}}_k||F^{\check{a}_k}_k(\mathcal{V}^{\Re,\pi,\check{a}}_k) - {}^1F^{\check{a}_k}_k({}^1\mathcal{V}^{\Re,\pi,\check{a}}_k)|h_k\right].$$

Since $F = {}^1F \vee {}^2F$ and ${}^1F$ is a Lipschitz function, we also get

$$|F^{\check{a}_k}_k(\mathcal{V}^{\Re,\pi,\check{a}}_k) - {}^1F^{\check{a}_k}_k({}^1\mathcal{V}^{\Re,\pi,\check{a}}_k)| \leqslant |\delta F_k|_\infty + C|\mathcal{V}^{\Re,\pi,\check{a}}_k - {}^1\mathcal{V}^{\Re,\pi,\check{a}}_k|,$$

and then, by using Young's inequality and discrete Gronwall's lemma, we deduce from the last and the penultimate inequalities that

$$\mathbb{E}\left[|\mathcal{U}^{\Re,\pi,\check{a}}_i - {}^1\mathcal{U}^{\Re,\pi,\check{a}}_i|^2\right] \leqslant C\left(\mathbb{E}[|\delta\xi|^2 + \sum_{k=i}^{n-1}|\delta F_k|^2_\infty h_k + \mathbb{E}[|\mathcal{A}^{\check{a}}_n - {}^1\mathcal{A}^{\check{a}}_n|^2 + |\mathcal{A}^{\check{a}}_i - {}^1\mathcal{A}^{\check{a}}_i|^2]\right). \tag{3.24}$$

Moreover we compute, for all $k \in [\![i,n]\!]$,

$$\mathbb{E}[|\mathcal{A}^{\check{a}}_k - {}^1\mathcal{A}^{\check{a}}_k|^2] \leqslant \mathbb{E}[|\mathcal{N}^{\check{a}}|^2 \sup_{0\leqslant m\leqslant n, t_m\in\Re}|\delta C_{t_m}|^2_\infty].$$

If $p = 2$, then $\mathcal{N}^{\check{a}} \leqslant \kappa$ yields

$$\mathbb{E}[|\mathcal{A}^{\check{a}}_k - {}^1\mathcal{A}^{\check{a}}_k|^2] \leqslant \kappa^2\mathbb{E}[\sup_{0\leqslant m\leqslant n, t_m\in\Re}|\delta C_{t_m}|^2_\infty].$$

Otherwise, from Proposition 3.4, Hölder inequality and the fact that $\mathcal{N}^{\check{a}} \leqslant \kappa$, we deduce

$$\begin{aligned}\mathbb{E}[|\mathcal{A}^{\check{a}}_k - {}^1\mathcal{A}^{\check{a}}_k|^2] &\leqslant \mathbb{E}\left[|\mathcal{N}^{\check{a}}|^{\frac{2p}{p-2}}\right]^{\frac{p-2}{p}}\mathbb{E}\left[\sup_{0\leqslant m\leqslant n, t_m\in\Re}|\delta C_{t_m}|^p_\infty\right]^{2/p}\\ &\leqslant \mathbb{E}\left[\kappa^{\frac{2p}{p-2}-2}|\mathcal{N}^{\check{a}}|^2\right]^{\frac{p-2}{p}}\mathbb{E}\left[\sup_{0\leqslant m\leqslant n, t_m\in\Re}|\delta C_{t_m}|^p_\infty\right]^{2/p}\\ &\leqslant C_p\kappa^{4/p}\mathbb{E}\left[\sup_{0\leqslant m\leqslant n, t_m\in\Re}|\delta C_{t_m}|^p_\infty\right]^{2/p}.\end{aligned}$$

Inserting the last estimate into (3.24), we get

$$\mathbb{E}\left[|\mathcal{U}_i^{\Re,\pi,\check{a}} - {}^1\mathcal{U}_i^{\Re,\pi,\check{a}}|^2\right] \leqslant C\left(\mathbb{E}[|\delta\xi|^2 + \sum_{k=i}^{n-1}|\delta F_k|_\infty^2\, h_k\right) + C_p\kappa^{4/p}\mathbb{E}\left[\sup_{0\leqslant m\leqslant n, t_m\in\Re}|\delta C_{t_m}|_\infty^p\right]^{2/p}.$$

By symmetry, we have the same estimate for $\mathbb{E}\left[|\mathcal{U}_i^{\Re,\pi,\check{a}} - {}^2\mathcal{U}_i^{\Re,\pi,\check{a}}|^2\right]$. Therefore, from (3.23) and the fact that j is arbitrary, we deduce the wanted estimate for $\mathbb{E}[|\delta\widetilde{\mathcal{Y}}_i^{\Re,\pi}|^2]$. The estimate for $\mathbb{E}[|\delta\mathcal{Y}_i^{\Re,\pi}|^2]$ is a direct corollary.

Step 3. Stability of the Z component. Observing that $\delta\mathcal{Z}_k^{\Re,\pi} = \mathbb{E}_{t_k}[(\delta\mathcal{Y}_{k+1}^{\Re,\pi} - \mathbb{E}_{t_k}[\delta\mathcal{Y}_{k+1}^{\Re,\pi}])H_k^\top]$, one computes

$$h_k|\delta\mathcal{Z}_k^{\Re,\pi}|^2 \leqslant C\mathbb{E}_{t_k}\Big[|\delta\mathcal{Y}_{k+1}^{\Re,\pi}|^2 - |\mathbb{E}_{t_k}\Big[\delta\mathcal{Y}_{k+1}^{\Re,\pi}\Big]|^2\Big] \tag{3.25}$$

From the scheme's definition, we have

$$|\mathbb{E}_{t_k}\Big[\delta\mathcal{Y}_{k+1}^{\Re,\pi}\Big]|^2 \geqslant |\delta\widetilde{\mathcal{Y}}_k^{\Re,\pi}|^2 - 2|\delta\widetilde{\mathcal{Y}}_k^{\Re,\pi}[{}^1F_k({}^1\mathcal{Z}_k^{\Re,\pi}) - {}^2F_k({}^2\mathcal{Z}_k^{\Re,\pi})]h_k|\,.$$

Inserting the last estimate into (3.25) and using (HFd_p), we obtain, for some $\eta > 0$,

$$h_k|\delta\mathcal{Z}_k^{\Re,\pi}|^2 \leqslant C\left(\mathbb{E}_{t_k}\Big[|\delta\mathcal{Y}_{k+1}^{\Re,\pi}|^2\Big] - |\delta\widetilde{\mathcal{Y}}_k^{\Re,\pi}|^2 + Ch_k\left(1+\frac{1}{\eta}\right)|\delta\widetilde{\mathcal{Y}}_k^{\Re,\pi}|^2 + \eta h_k|\delta\mathcal{Z}_k^{\Re,\pi}|^2 + h_k|\delta F_k|_\infty^2\right).$$

Taking expectation on both sides and summing over k with $\eta = 1/2$, we get

$$\begin{aligned}
&\frac{1}{2}\sum_{k=i}^{n-1}h_k\mathbb{E}\Big[|\delta\mathcal{Z}_k^{\Re,\pi}|^2\Big]\\
&\leqslant C\left(\mathbb{E}\Big[|\delta\mathcal{Y}_n^{\Re,\pi}|^2\Big] + \sum_{\substack{k=i\\ t_k\in\Re}}^{n-1}\mathbb{E}\Big[|\delta\mathcal{Y}_k^{\Re,\pi}|^2 - |\delta\widetilde{\mathcal{Y}}_k^{\Re,\pi}|^2\Big] + \max_{i\leqslant k\leqslant n-1}\mathbb{E}\Big[|\delta\widetilde{\mathcal{Y}}_k^{\Re,\pi}|^2\Big] + \sum_{k=i}^{n-1}h_k|\delta F_k|_\infty^2\right),\\
&\leqslant C\left(\mathbb{E}\Big[|\delta\mathcal{Y}_n^{\Re,\pi}|^2\Big] + \kappa\max_{i\leqslant k\leqslant n-1}\mathbb{E}\Big[|\delta\mathcal{Y}_k^{\Re,\pi}|^2 + |\delta\widetilde{\mathcal{Y}}_k^{\Re,\pi}|^2\Big] + \sum_{k=i}^{n-1}h_k|\delta F_k|_\infty^2\right).
\end{aligned}$$

The proof is concluded using estimates on $\delta\widetilde{\mathcal{Y}}^{\Re,\pi}$ and $\delta\mathcal{Y}^{\Re,\pi}$ already obtained in the first part of the proof. □

We will now use this general stability result on obliquely reflected backward schemes to obtain a L^2-stability result for the scheme (1.4) (see [6] for a general definition of L^2-stability for backward schemes). Firstly, we introduce a perturbed version of the scheme given in (1.4).

Definition 3.2. *(i) The terminal condition is given by a $\mathcal{F}_T$-measurable random variable $\bar{Y}_n \in \mathscr{L}^2$;*

(ii) for $0 \leqslant i < n$,

$$\begin{cases} \bar{Z}_i^{\Re,\pi} := \mathbb{E}[\bar{Y}_{i+1}^{\Re,\pi} H_i \mid \mathcal{F}_{t_i}], \\ \tilde{\bar{Y}}_i^{\Re,\pi} := \mathbb{E}[\bar{Y}_{i+1}^{\Re,\pi} \mid \mathcal{F}_{t_i}] + h_i f(X_{t_i}^\pi, \tilde{\bar{Y}}_i^{\Re,\pi}, \bar{Z}_i^{\Re,\pi}) + \zeta_i^f, \\ \bar{Y}_i^{\Re,\pi} := \tilde{\bar{Y}}_i^{\Re,\pi} \mathbf{1}_{\{t_i \notin \Re\}} + \bar{\mathcal{P}}_{t_i}(X_{t_i}^\pi, \tilde{\bar{Y}}_i^{\Re,\pi}) \mathbf{1}_{\{t_i \in \Re\}}, \end{cases} \tag{3.26}$$

with $\bar{\mathcal{P}}$ *the oblique projection*

$$\bar{\mathcal{P}}_{t_i} : (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \left(\max_{j \in \mathcal{I}} \{ y^j - \bar{c}_{t_i}^{kj}(x) \} \right)_{1 \leqslant k \leqslant d},$$

associated to costs $\bar{c}_{t_i}(x) := c(x) + \zeta_{t_i}^c$. *Perturbations* $\zeta_i^Y := (\zeta_i^f, \zeta_{t_i}^c)$ *are* $\mathcal{F}_{t_i}$-*measurable and square integrable random variables. Moreover we assume that the random costs* $(\bar{c}_{t_i}(X_{t_i}))_{0 \leqslant i \leqslant n}$ *satisfy the structure conditions* (2.1).

Setting $\delta Y_i = Y_i^{\Re,\pi} - \bar{Y}_i^{\Re,\pi}$, $\delta \widetilde{Y}_i = \widetilde{Y}_i^{\Re,\pi} - \tilde{\bar{Y}}_i^{\Re,\pi}$ and $\delta Z_i = Z_i^{\Re,\pi} - \bar{Z}_i^{\Re,\pi}$, we obtain the following L^2-stability result for the scheme (1.4).

Proposition 3.6. *Assume that* (Hf) *is in force and, for all* $p \geqslant 2$,

$$\mathbb{E}\left[|\bar{Y}_n|^2 + \sum_{i=0}^{n-1} |\zeta_i^f|^2 + \sup_{0 \leqslant i \leqslant n} |\zeta_{t_i}^c|^p \right] \leqslant C. \tag{3.27}$$

We also assume that $|\pi| L^Y < 1$ *and*

$$\left(\sup_{0 \leqslant i \leqslant n-1} h_i |H_i| \right) L^Z \leqslant 1. \tag{3.28}$$

Then schemes (1.4) *and* (3.26) *are well defined and the following* L^2-*stability holds true, for all* $p \geqslant 2$,

$$\begin{aligned} & \sup_{0 \leqslant i \leqslant n} \mathbb{E}\Big[|\delta Y_i|^2 + |\delta \widetilde{Y}_i|^2 \Big] + \frac{1}{\kappa} \sum_{i=0}^{n-1} h_i \mathbb{E}\big[|\delta Z_i|^2 \big] \\ \leqslant \; & C \left(\mathbb{E}\big[|\delta Y_n|^2 \big] + \sum_{i=0}^{n-1} \frac{1}{h_i} \mathbb{E}\big[|\zeta_i^f|^2 \big] \right) + C_p \kappa^{4/p} \mathbb{E}\left[\sup_{\substack{0 \leqslant i \leqslant n \\ t_i \in \Re}} |\zeta_i^c|^p \right]^{2/p}. \end{aligned} \tag{3.29}$$

Proof. Since we have assumed $|\pi| L^Y < 1$, then a simple fixed point argument shows that schemes (1.4) and (3.26) are well defined, i.e. there exists a unique solution to each scheme.

For the L^2-stability, we want to apply Proposition 3.5 with ${}^1\xi = g(X_T^\pi)$, ${}^2\xi = \bar{Y}_n$, ${}^1F_i(z) = f(X_{t_i}^\pi, \widetilde{Y}_i^{\Re,\pi}, z)$, ${}^2F_i(z) = f(X_{t_i}^\pi, \tilde{\bar{Y}}_i^{\Re,\pi}, z) + \zeta_i^f$, ${}^1C_{t_i} = c(X_{t_i}^\pi)$ and ${}^2C_{t_i} = c(X_{t_i}^\pi) + \zeta_{t_i}^c$. To do this, we have to check that assumption (HFd_p) is fulfilled for these two obliquely reflected backward schemes. Firstly, we have assumed

$$\left(\sup_{0 \leqslant i \leqslant n-1} h_i |H_i| \right) L^Z \leqslant 1.$$

Moreover, hypothesis (Hf), assumption (3.27) and classical estimates for processes X and X^π leads to

$$\mathbb{E}\Bigg[|{}^1\xi|^2+|{}^2\xi|^2+\sum_{i=0}^{n-1}[|{}^1F_i(0)|^2+|{}^2F_i(0)|^2]h_i+\sup_{0\leqslant i\leqslant n}[|{}^1C_{t_i}|^p+|{}^2C_{t_i}|^p]\Bigg]$$
$$\leqslant C_p+C\mathbb{E}\Bigg[\sup_{0\leqslant i\leqslant n}[|\widetilde{Y}_i^{\Re,\pi}|^2+|\widetilde{\widetilde{Y}}_i^{\Re,\pi}|^2]\Bigg].$$

To estimate quantities $\mathbb{E}\Big[\sup_{0\leqslant i\leqslant n}|\widetilde{Y}_i^{\Re,\pi}|^2\Big]$ and $\mathbb{E}\Big[\sup_{0\leqslant i\leqslant n}|\widetilde{\widetilde{Y}}_i^{\Re,\pi}|^2\Big]$, we just have to rewrite slightly the first step of the proof of Proposition 3.2. The beginning of the proof stays true: (3.7) yields, for all $i\in[\![0,n]\!]$,

$$\begin{aligned}\mathbb{E}\Bigg[\sup_{i\leqslant k\leqslant n}\Big[|\widetilde{Y}_k^{\Re,\pi}|^2+|\widetilde{\widetilde{Y}}_k^{\Re,\pi}|^2\Big]\Bigg]&\leqslant C\mathbb{E}\Bigg[|{}^1\xi|^2+|{}^2\xi|^2+\sum_{k=i}^{n-1}\Big[|{}^1F_k(0)|^2+|{}^2F_k(0)|^2\Big]h_k\Bigg]\\&\leqslant C\left(1+\sum_{k=i}^{n-1}\mathbb{E}\Bigg[\sup_{k\leqslant m\leqslant n}\Big[|\widetilde{Y}_m^{\Re,\pi}|^2+|\widetilde{\widetilde{Y}}_m^{\Re,\pi}|^2\Big]\Bigg]h_k\right).\end{aligned}$$

Thus, the discrete Gronwall lemma allows to conclude that

$$\mathbb{E}\Bigg[\sup_{0\leqslant k\leqslant n}\Big[|\widetilde{Y}_k^{\Re,\pi}|^2+|\widetilde{\widetilde{Y}}_k^{\Re,\pi}|^2\Big]\Bigg]\leqslant C$$

and then assumption (HFd_p) is fulfilled. Proposition 3.5 and (Hf) imply, for all $i\in[\![0,n]\!]$,

$$\begin{aligned}&\sup_{i\leqslant k\leqslant n}\mathbb{E}\Big[|\delta Y_k|^2+|\delta\widetilde{Y}_k|^2\Big]+\frac{1}{\kappa}\sum_{k=i}^{n-1}h_k\mathbb{E}\Big[|\delta Z_k|^2\Big]\\&\leqslant C\left(\mathbb{E}\Big[|\delta Y_n|^2\Big]+\sum_{k=i}^{n-1}|\delta F_k|_\infty^2\right)+C_p\kappa^{4/p}\mathbb{E}\Bigg[\sup_{\substack{0\leqslant k\leqslant n\\ t_k\in\Re}}|\zeta^c_{t_k}|^p\Bigg]^{2/p}\\&\leqslant C\left(\mathbb{E}\Big[|\delta Y_n|^2\Big]+\sum_{k=0}^{n-1}\frac{1}{h_k}\mathbb{E}\Big[|\zeta^f_k|^2\Big]+\sum_{k=i}^{n-1}\sup_{k\leqslant m\leqslant n}\mathbb{E}\Big[|\delta Y_m|^2+|\delta\widetilde{Y}_m|^2\Big]h_k\right)+C_p\kappa^{4/p}\mathbb{E}\Bigg[\sup_{\substack{0\leqslant k\leqslant n\\ t_k\in\Re}}|\zeta^c_{t_k}|^p\Bigg]^{2/p}.\end{aligned}$$

Applying the discrete Gronwall lemma to the last inequality completes the proof. □

3.4 Convergence analysis of the discrete-time approximation

We will give now the main result of this section that provides an upper bound for the error between the obliquely reflected backward scheme (1.4) and the discretely obliquely reflected BSDE (1.3).

Theorem 3.1. *Assume that (Hf) is in force. We also assume that $|\pi|L^Y < 1$ and weights $(H_i)_{0\leqslant i\leqslant n-1}$ are given by*

$$(H_i)^\ell = \frac{-R}{h_i} \vee \frac{W^\ell_{t_{i+1}} - W^\ell_{t_i}}{h_i} \wedge \frac{R}{h_i}, \quad 1 \leqslant \ell \leqslant d, \tag{3.30}$$

with R a positive parameter such that $RL^Z \leqslant 1$. Then the following holds:

$$\sup_{0\leqslant i\leqslant n} \mathbb{E}\Big[|\widetilde{Y}^{\Re}_{t_i} - \widetilde{Y}^{\Re,\pi}_i|^2 + |Y^{\Re}_{t_i} - Y^{\Re,\pi}_i|^2\Big] + \frac{1}{\kappa}\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |Z^{\Re}_s - Z^{\Re,\pi}_i|^2 ds\right] \leqslant C_R\left(|\pi|^{1/2} + \kappa|\pi|\right).$$

Proof.

Step 1. Expression of the perturbing error. Since we want to apply Proposition 3.6, we first observe that $(Y^{\Re}, Z^{\Re})$ can be rewritten as a perturbed obliquely reflected backward scheme. Namely, setting $\bar{Y}_i := Y^{\Re}_{t_i}$ and $\tilde{\bar{Y}}_i := \widetilde{Y}^{\Re}_{t_i}$, for all $i \in [\![0, n]\!]$, we have

$$\begin{cases} \bar{Z}_i := \mathbb{E}[\bar{Y}_{i+1}H_i \mid \mathcal{F}_{t_i}], \\ \tilde{\bar{Y}}_i := \mathbb{E}[\bar{Y}_{i+1} \mid \mathcal{F}_{t_i}] + h_i f(X^\pi_{t_i}, \tilde{\bar{Y}}_i, \bar{Z}_i) + \zeta^f_i, \\ \bar{Y}_i := \tilde{\bar{Y}}_i \mathbf{1}_{\{t_i \notin \Re\}} + \bar{\mathcal{P}}_{t_i}(X^\pi_{t_i}, \tilde{\bar{Y}}_i)\mathbf{1}_{\{t_i \in \Re\}}, \end{cases} \tag{3.31}$$

with

$$\zeta^f_i = \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} \Big(f(X_s, \widetilde{Y}^{\Re}_s, Z^{\Re}_s) - f(X^\pi_{t_i}, \widetilde{Y}^{\Re}_{t_i}, \bar{Z}_i)\Big) ds\right] \quad \text{and} \quad \zeta^c_{t_i} = c(X_{t_i}) - c(X^\pi_{t_i}).$$

Let us check that (3.27) is fulfilled for all $p \geqslant 2$: using (Hf), Proposition 2.3 and classical estimates for X and X^π, we get

$$\begin{aligned} &\mathbb{E}\left[|\bar{Y}_n|^2 + \sum_{i=0}^{n-1}|\zeta^f_i|^2 + \sup_{0\leqslant i\leqslant n}|\zeta^c_{t_i}|^p\right] \\ &\leqslant C_p\mathbb{E}\left[1 + \sup_{s\in[0,T]}|X_s|^p + \sup_{i\in[\![0,n]\!]}|X^\pi_{t_i}|^p + \sup_{s\in[0,T]}|\widetilde{Y}^{\Re}_s|^2 + \int_0^T |Z^{\Re}_s|^2 ds + \sum_{i=0}^{n-1}|\widetilde{Y}^{\Re}_{t_{i+1}}H_i h_i|^2\right] \\ &\leqslant C_p\left(1 + \mathbb{E}\left[\sup_{s\in[0,T]}|\widetilde{Y}^{\Re}_s|^4 + \left(\sum_{i=0}^{n-1}|H_i h_i|^2\right)^2\right]\right). \end{aligned}$$

Applying Burkholder-Davis-Gundy inequality, we have $\mathbb{E}\Big[\Big(\sum_{i=0}^{n-1}|H_i h_i|^2\Big)^2\Big] \leqslant C$ and so (3.27) is fulfilled. Finally, we easily check that (3.30) implies (3.28).

Step 2. Discretization error for the Y component. Setting $p = 4$, we apply Proposition 3.6 and get by direct calculations

$$
\begin{aligned}
&\sup_{0\leqslant i\leqslant n} \mathbb{E}\Big[|\widetilde{Y}^{\Re}_{t_i} - \widetilde{Y}^{\Re,\pi}_i|^2 + |Y^{\Re}_{t_i} - Y^{\Re,\pi}_i|^2\Big] \\
&\leqslant C\left(\mathbb{E}\big[|g(X_T) - g(X^{\pi}_T)|^2\big] + \sum_{i=0}^{n-1}\frac{1}{h_i}\mathbb{E}\Big[|\zeta^f_i|^2\Big]\right) + C_p\kappa\mathbb{E}\left[\sup_{\substack{0\leqslant i\leqslant n\\ t_i\in\Re}}|\zeta^c_{t_i}|^4\right]^{1/2} \\
&\leqslant C\mathbb{E}\big[|X_T - X^{\pi}_T|^2\big] + C\sup_{0\leqslant i\leqslant n-1}\mathbb{E}\left[\sup_{s\in[t_i,t_{i+1}]}|X_s - X^{\pi}_{t_i}|^2\right] + C\mathbb{E}\left[\int_0^T |\widetilde{Y}^{\Re}_s - \widetilde{Y}^{\Re}_{\pi(s)}|^2 ds\right] \\
&\quad + C\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|Z^{\Re}_s - \bar{Z}^{\Re}_{t_i}|^2 ds\right] + C\mathbb{E}\left[\sum_{i=0}^{n-1}|\bar{Z}^{\Re}_{t_i} - \bar{Z}_i|^2 h_i\right] + C\kappa\mathbb{E}\Big[\sup_{0\leqslant i\leqslant n-1}|X_{t_i} - X^{\pi}_{t_i}|^4\Big]^{1/2}.
\end{aligned}
$$

Classical estimations on the Euler scheme for SDEs, see e.g. [18], yield

$$
\mathbb{E}\big[|X_T - X^{\pi}_T|^2\big] + \sup_{0\leqslant i\leqslant n-1}\mathbb{E}\left[\sup_{s\in[t_i,t_{i+1}]}|X_s - X^{\pi}_{t_i}|^2\right] + \kappa\mathbb{E}\Big[\sup_{0\leqslant i\leqslant n-1}|X_{t_i} - X^{\pi}_{t_i}|^4\Big]^{1/2} \leqslant C\kappa|\pi|.
$$

Applying Proposition 2.5 and Proposition 2.6, we obtain

$$
\mathbb{E}\left[\int_0^T |\widetilde{Y}^{\Re}_s - \widetilde{Y}^{\Re}_{\pi(s)}|^2 ds\right] + \mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|Z^{\Re}_s - \bar{Z}^{\Re}_{t_i}|^2 ds\right] \leqslant C(|\pi| + |\pi|^{1/2} + \kappa|\pi|).
$$

It remains to bound the term:

$$
\mathbb{E}\left[\sum_{i=0}^{n-1}|\bar{Z}^{\Re}_{t_i} - \bar{Z}_i|^2 h_i\right] \leqslant 2\underbrace{\mathbb{E}\left[\sum_{i=0}^{n-1}\left|\bar{Z}^{\Re}_{t_i} - \mathbb{E}_{t_i}\left[Y^{\Re}_{i+1}\frac{\Delta W_i}{h_i}\right]\right|^2 h_i\right]}_{:=A} + 2\underbrace{\mathbb{E}\left[\sum_{i=0}^{n-1}\left|\mathbb{E}_{t_i}\left[Y^{\Re}_{i+1}\left(\frac{\Delta W_i}{h_i} - H_i\right)\right]\right|^2 h_i\right]}_{:=B}.
$$

By remarking that $\bar{Z}^{\Re}_{t_i} = \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z^{\Re}_s dW_s \frac{\Delta W_i}{h_i}\right]$, we have

$$
\begin{aligned}
A &= \mathbb{E}\left[\sum_{i=0}^{n-1}\left|\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} f(X_s, Y^{\Re}_s, Z^{\Re}_s)ds\frac{\Delta W_i}{h_i}\right]\right|^2 h_i\right] \\
&\leqslant \mathbb{E}\left[\sum_{i=0}^{n-1} h_i\int_{t_i}^{t_{i+1}}|f(X_s, Y^{\Re}_s, Z^{\Re}_s)|^2 ds\right] \leqslant |\pi|\mathbb{E}\left[\int_0^T |f(X_s, Y^{\Re}_s, Z^{\Re}_s)|^2 ds\right] \leqslant C|\pi|.
\end{aligned}
$$

Finally, we also get by standard calculations, Proposition 2.3 and classical results about Gaussian distribution tails

$$
\begin{aligned}
B &\leqslant \sup_{0\leqslant i\leqslant n-1}\mathbb{E}\big[|Y_{t_{i+1}}|^2\big] \times \sup_{0\leqslant i\leqslant n-1}\mathbb{E}\left[\left|\frac{\Delta W_i}{h_i} - H_i\right|^2\right] \leqslant C\left(\frac{2d}{h_i}\int_{Rh_i^{-1}}^{+\infty} x^2\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx\right) \\
&\leqslant \frac{C}{h_i}\left(Rh_i^{-1}e^{-R^2h_i^{-2}/2}\right) \leqslant \frac{CR}{h_i^2}\left(\frac{2h_i^2}{R^2}\right)^{3/2} \leqslant C_R|\pi|.
\end{aligned}
$$

Step 3. Discretization error for the Z component. Let us remark that we have

$$\frac{1}{\kappa}\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|Z_s^{\Re}-Z_i^{\Re,\pi}|^2ds\right] \leqslant \frac{1}{\kappa}\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|Z_s^{\Re}-\bar{Z}_{t_i}^{\Re}|^2ds\right]+\frac{1}{\kappa}\mathbb{E}\left[\sum_{i=0}^{n-1}|\bar{Z}_{t_i}^{\Re}-\bar{Z}_i|^2h_i\right] + \frac{1}{\kappa}\mathbb{E}\left[\sum_{i=0}^{n-1}|\bar{Z}_i-Z_i^{\Re,\pi}|^2h_i\right].$$

Previous calculations already yield

$$\frac{1}{\kappa}\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|Z_s^{\Re}-\bar{Z}_{t_i}^{\Re}|^2ds\right]+\frac{1}{\kappa}\mathbb{E}\left[\sum_{i=0}^{n-1}|\bar{Z}_{t_i}^{\Re}-\bar{Z}_i|^2h_i\right] \leqslant \frac{C}{\kappa}\left(|\pi|^{1/2}+\kappa|\pi|\right).$$

Moreover, we apply Proposition 3.6 to obtain

$$\frac{1}{\kappa}\mathbb{E}\left[\sum_{i=0}^{n-1}|\bar{Z}_i-Z_i^{\Re,\pi}|^2h_i\right] \leqslant C\left(|\pi|^{1/2}+\kappa|\pi|\right),$$

thanks to estimates obtained in step 2. □

4 Application to continuously reflected BSDEs

This section is devoted to the study of the error between the scheme (1.4) and the continuously obliquely reflected BSDEs (1.1). An upper bound of this error is stated in Subsection 4.2 while Subsection 4.1 is devoted to the error between the continuously obliquely reflected BSDEs (1.1) and the discretely obliquely reflected BSDEs (1.3). Before these results, we start by giving some classical estimates on the solution of (1.1).

Proposition 4.1. *Assume that (Hf) is in force. There exists a unique solution $(Y,Z,K) \in \mathscr{S}_2 \times \mathscr{H}_2 \times \mathscr{K}_2$ to (1.1) and it satisfies, for all $p \geqslant 2$,*

$$|Y|_{\mathscr{S}_p}+|Z|_{\mathscr{H}_p}+|K_T|_{\mathscr{L}^p} \leqslant C_p.$$

Proof. The existence and uniqueness result comes from [7]. Concerning estimates, we want to apply Proposition 2.1 with terminal condition $\xi = g(X_T)$, random generator $F(s,z) = f(X_s,Y_s,z)$ and costs $C_s^{ij} = c^{ij}(X_s)$. So, we just have to show that (HF_p) is in force. Thus, using the fact that f is a Lipschitz function with respect to y, it is sufficient to control $\widetilde{Y}^{\Re}$ in $\mathscr{S}^p$ to conclude. We are able to obtain estimates on $|\widetilde{Y}^{\Re}|_{\mathscr{S}^p}$ by a direct adaptation to the continuous time setting of the proof of Proposition 2.3. □

4.1 Error between discretely and continuously reflected BSDEs

We show here that the error between the continuously reflected BSDE (1.1) and the discretely reflected BSDE (1.3) is controlled in a convenient way. We start by introducing a temporary assumption.

(Hz) For all $(x,y,z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}^{d,d}$, $|f(x,y,z)| \leqslant C(1+|x|+|y|)$.

Proposition 4.2. *Assume that (Hf) and (Hz) are in force, then*

$$\mathbb{E}\left[\sup_{r\in\Re}|Y_r - Y_r^{\Re}|^2 + \sup_{t\in[0,T]}|Y_t - \tilde{Y}_t^{\Re}|^2\right] \leqslant C|\Re|\log(2T/|\Re|).$$

Moreover, if the cost functions are constant, we obtain a better rate of convergence, namely,

$$\mathbb{E}\left[\sup_{r\in\Re}|Y_r - Y_r^{\Re}|^2 + \sup_{t\in[0,T]}|Y_t - \tilde{Y}_t^{\Re}|^2\right] \leqslant C|\Re|.$$

Proof. 1. We denote $(\check{Y}, \check{Z}, \check{K})$ the solution of an auxiliary continuously obliquely reflected BSDE with cost functions c, with terminal condition $\xi := g(X_T)$ and whose random generator is given by

$$\check{f}(s, z) = f(X_s, Y_s, z) \vee f(X_s, \tilde{Y}_s^{\Re}, z)\ .$$

We also denote $(\tilde{Y}, \tilde{Z}, \tilde{K})$ the solution of the continuously obliquely reflected BSDE with cost functions c, with terminal condition $\xi := g(X_T)$ and with random driver $\tilde{f}(s, z) = f(X_s, \tilde{Y}_s^{\Re}, z)$. From Proposition 2.2, we know that each component of $\check{Y}$, Y and $\tilde{Y}$ can be represented as optimal values of some control problem namely

$$(\check{Y}_t)^i = \operatorname*{ess\,sup}_{a\in\mathscr{A}_{i,t}} \check{U}_t^a = \check{U}^{\check{a}_t}, \quad (Y_t)^i = \operatorname*{ess\,sup}_{a\in\mathscr{A}_{i,t}} U_t^a, \quad (\tilde{Y}_t)^i = \operatorname*{ess\,sup}_{a\in\mathscr{A}_{i,t}} U_t^{\Re,a}, \tag{4.1}$$

with $t \in [0, T]$, $i \in \mathcal{I}$, $\check{U}^a$, U^a and $U^{\Re,a}$ solutions to following "switched" BSDEs:

$$\check{U}_t^a = \xi^{a_T} + \int_t^T \check{f}^{a_s}(s, \check{V}_s^a)\mathrm{d}s - \int_t^T \check{V}_s^a \mathrm{d}W_s - A_T^a + A_t^a\ , \tag{4.2}$$

$$U_t^a = \xi^{a_T} + \int_t^T f^{a_s}(X_s, Y_s, V_s^a)\mathrm{d}s - \int_t^T V_s^a \mathrm{d}W_s - A_T^a + A_t^a\ , \tag{4.3}$$

$$U_t^{\Re,a} = \xi^{a_T} + \int_t^T f^{a_s}(X_s, Y_s^{\Re}, V_s^{\Re,a})\mathrm{d}s - \int_t^T V_s^{\Re,a} \mathrm{d}W_s - A_T^a + A_t^a\ , \tag{4.4}$$

and $\check{a}$ the optimal strategy given by Proposition 2.2. Using a comparison argument, we easily check that $\check{U}^a \geqslant U^a \vee U^{\Re,a}$, for any strategy $a \in \mathscr{A}_{i,t}$. This estimate combined with (4.1) leads to

$$\check{Y}^\ell \geqslant Y^\ell \vee (\tilde{Y})^\ell \ \text{ for all } \ \ell \in \{1, \ldots, d\}\ .$$

Moreover, Corollary 2.1 and (4.1) give us that

$$(\tilde{Y}_t^{\Re})^i = \operatorname*{ess\,sup}_{a\in\mathscr{A}_{i,t}^{\Re}} U_t^{\Re,a} \leqslant \operatorname*{ess\,sup}_{a\in\mathscr{A}_{i,t}} U_t^{\Re,a} = (\tilde{Y}_t)^i.$$

Then, we finally obtain

$$\check{Y}^\ell \geqslant Y^\ell \vee (\tilde{Y}^{\Re})^\ell \ \text{ for all } \ \ell \in \{1, \ldots, d\}\ . \tag{4.5}$$

Furthermore we observe that, for all $\ell \in \{1, \dots, d\}$ and all $t \in [0, T]$,

$$|(Y_t)^\ell - (\tilde{Y}_t^{\Re})^\ell| \leqslant |(\check{Y}_t)^\ell - (Y_t)^\ell| + |(\check{Y}_t)^\ell - (\tilde{Y}_t^{\Re})^\ell|. \tag{4.6}$$

We will now deal separately with the two term in the right hand side of the above inequality.

2.a We start by studying the first term. From the representation in terms of switched BSDEs given in (4.1), we know that $(\check{Y}_t)^\ell = \check{U}_t^{\check{a}}$ and $(Y_t)^\ell \geqslant U_t^{\check{a}}$ with U^a solution to (4.3). Indeed $\check{a} \in \mathscr{A}_{\ell,t}$ is the optimal strategy associated to the driver $\check{f}$ and is *a priori* sub-optimal for the driver f. Combining this with (4.5), we obtain that

$$0 \leqslant (\check{Y}_t)^\ell - (Y_t)^\ell \leqslant \check{U}_t^{\check{a}} - U_t^{\check{a}}$$

and we only need now to control the right hand inequality. By applying Itô's formula to the process $e^{\beta t}|\check{U}_t^{\check{a}} - U_t^{\check{a}}|^2$ and by using assumption (Hf), usual computations lead to, for some $\beta > 0$,

$$\begin{aligned}
&e^{\beta t}|\check{U}_t^{\check{a}} - U_t^{\check{a}}|^2 + \mathbb{E}_t\left[\int_t^T e^{\beta s}|\check{V}_s^{\check{a}} - V_s^{\check{a}}|^2 ds\right] \\
\leqslant &\mathbb{E}_t\left[\int_t^T e^{\beta s}\left[2C|\check{U}_s^{\check{a}} - U_s^{\check{a}}|(|\check{V}_s^{\check{a}} - V_s^{\check{a}}| + |Y_s - \tilde{Y}_s^{\Re}|) - \beta|\check{U}_s^{\check{a}} - U_s^{\check{a}}|^2\right] ds\right] \\
\leqslant &\mathbb{E}_t\left[\int_t^T e^{\beta s}\left[(2C^2 - \beta)|\check{U}_s^{\check{a}} - U_s^{\check{a}}|^2 + |\check{V}_s^{\check{a}} - V_s^{\check{a}}|^2 + |Y_s - \tilde{Y}_s^{\Re}|^2\right] ds\right],
\end{aligned}$$

and then, for any β large enough,

$$e^{\beta t}|(\check{Y}_t)^\ell - (Y_t)^\ell|^2 \leqslant \mathbb{E}_t\left[\int_t^T e^{\beta_s}|Y_s - \tilde{Y}_s^{\Re}|^2 \mathrm{d}s\right]. \tag{4.7}$$

2.b We now study the second term in the right hand side of (4.6). Combining (4.5) and the representation in term of "switched BSDEs" given by (4.1), we have, for all $t \in [0, T]$, $\ell \in \{1, \dots, d\}$,

$$0 \leqslant (\check{Y}_t)^\ell - (\tilde{Y}_t^{\Re})^\ell \leqslant \check{U}_t^{\check{a}} - (\tilde{Y}_t^{\Re})^\ell \tag{4.8}$$

for some $\check{a} \in \mathscr{A}_{t,\ell}$. We now introduce the strategy a, standing for the projection of $\check{a} = (\check{\theta}_k, \check{\alpha}_k)$ on the grid $\Re$, namely: $a := (\theta_k, \alpha_k) \in \mathscr{A}_{t,\ell}^{\Re}$ defined by

$$\theta_k = \inf\{r \geqslant \check{\theta}_k \ , \ r \in \Re\} \ \text{ and } \ \alpha_k = \check{\alpha}_k.$$

Note that, if the optimal strategy $\check{a}$ has many time of switching on $(r_j, r_{j+1}]$, where r_j, r_{j+1} belong to the grid $\Re$, the projected strategy a will have many instantaneous switches at r_{j+1}, see also Remark 2.2 .
From Corollary 2.1, we have $\tilde{Y}_t^{\Re} \geqslant U_t^{\Re,a}$ which, combined to (4.8), leads to

$$|(\check{Y}_t)^\ell - (\tilde{Y}_t^{\Re})^\ell| \leqslant |\check{U}_t^{\check{a}} - U_t^{\Re,a}|\,. \tag{4.9}$$

To study the right hand side of the above inequality, we treat separately the case $t \notin \Re$ from the case $t \in \Re$. Let us start by assuming that $t \notin \Re$: $t \in]r_j, r_{j+1}[$ with r_j, r_{j+1} in $\Re$. We introduce a slight modification $\check{a}^\varepsilon := (\theta_k^\varepsilon, \alpha_k^\varepsilon) \in \mathscr{A}_{t,\ell}$ of the optimal strategy $\check{a}$ defined by

$$\theta_1^\varepsilon = \check{\theta}_1 1_{\check{\theta}_1 > t} + \{(t+\varepsilon) \wedge \check{\theta}_2\} 1_{\check{\theta}_1 = t}, \qquad \theta_k^\varepsilon = \check{\theta}_k \quad \forall k \neq 1, \qquad \alpha_k^\varepsilon = \check{\alpha}_k \quad \forall k \geqslant 0,$$

with $\varepsilon \in]0, r_{j+1} - r_j[$ a parameter. Then (4.9) becomes

$$|(\check{Y}_t)^\ell - (\tilde{Y}_t^\Re)^\ell| \leqslant |\check{U}_t^{\check{a}} - \check{U}_t^{\check{a}^\varepsilon}| + |\check{U}_t^{\check{a}^\varepsilon} - U_t^{\Re,a}|. \tag{4.10}$$

We also introduce continuous processes $\check{\Gamma}^\varepsilon := \check{U}^{\check{a}^\varepsilon} - A^{\check{a}^\varepsilon}$ and $\Gamma = U^{\Re,a} - A^a$. We then have, for all $s \in [t, T]$,

$$\check{\Gamma}_s^\varepsilon - \Gamma_s = \check{\Gamma}_T^\varepsilon - \Gamma_T + \int_s^T \{\check{f}^{\check{a}^\varepsilon}(u, \check{V}_u^{\check{a}^\varepsilon}) - f^a(u, \tilde{Y}_u^\Re, V_u^{\Re,a})\} du - \int_s^T (\check{V}_u^{\check{a}^\varepsilon} - V_u^{\Re,a}) \mathrm{d}W_u \,.$$

By applying Itô's formula to the process $e^{\beta s}|\check{\Gamma}_s^\varepsilon - \Gamma_s|^2$ and by using assumption (Hf), usual computations lead to, for $\beta > 0$ large enough,

$$\begin{aligned}
&e^{\beta t}|\check{\Gamma}_t^\varepsilon - \Gamma_t|^2 \\
\leqslant &\mathbb{E}_t\left[\int_t^T e^{\beta s}\left[2C|\check{\Gamma}_s^\varepsilon - \Gamma_s|\{|\check{f}^{\check{a}^\varepsilon}(s, \check{V}_s^{\check{a}^\varepsilon}) - \check{f}^a(s, \check{V}_s^{\check{a}^\varepsilon})| + |\check{V}_s^{\check{a}^\varepsilon} - V_s^{\Re,a}| + |Y_s - \tilde{Y}_s^\Re|\}\right] ds\right] \\
&- \beta \mathbb{E}_t\left[\int_t^T e^{\beta s}\left[|\check{\Gamma}_s^\varepsilon - \Gamma_s|^2\right] ds\right] - \mathbb{E}_t\left[\int_t^T e^{\beta s}|\check{V}_s^{\check{a}^\varepsilon} - V_s^{\Re,a}|^2 ds\right] + \mathbb{E}_t\Big[e^{\beta T}|\check{\Gamma}_T^\varepsilon - \Gamma_T|^2\Big] \\
\leqslant &\mathbb{E}_t\left[e^{\beta T}|\check{\Gamma}_T^\varepsilon - \Gamma_T|^2 + Ce^{\beta T}\int_t^T |\check{f}^{\check{a}^\varepsilon}(s, \check{V}_s^{\check{a}^\varepsilon}) - \check{f}^a(s, \check{V}_s^{\check{a}^\varepsilon})|^2 \mathrm{d}s + \int_t^T e^{\beta s}|Y_s - \tilde{Y}_s^\Re|^2 \mathrm{d}s\right].
\end{aligned} \tag{4.11}$$

On one hand, using (Hz) we compute that

$$\begin{aligned}
\int_t^T |\check{f}^{\check{a}^\varepsilon}(s, \check{V}_s^{\check{a}^\varepsilon}) - \check{f}^a(s, \check{V}_s^{\check{a}^\varepsilon})|^2 \mathrm{d}s &= \int_t^T \left|\sum_{k=1}^{N^{\check{a}^\varepsilon}} \check{f}^{\check{a}^\varepsilon}(s, \check{V}_s^{\check{a}^\varepsilon})(\mathbf{1}_{\{\theta_{k-1}^\varepsilon \leqslant s < \theta_k^\varepsilon\}} - \mathbf{1}_{\{\theta_{k-1} \leqslant s < \theta_k\}})\right|^2 \mathrm{d}s \\
&\leqslant C|N^{\check{a}}|^2 \sup_{s \in [0,T]} (1 + |X_s|^2 + |Y_s|^2 + |\tilde{Y}_s^\Re|^2)|\Re|
\end{aligned} \tag{4.12}$$

since $N^{\check{a}^\varepsilon} = N^{\check{a}}$. On the other hand, by using (Hf) we obtain

$$|\check{\Gamma}_T^\varepsilon - \Gamma_T|^2 = |A_T^{\check{a}^\varepsilon} - A_T^a|^2 \leqslant C|N^{\check{a}}|^2 \sup_{1 \leqslant k \leqslant \kappa} \sup_{r \in [r_{k-1}, r_k]} |X_r - X_{r_k}|^2 \tag{4.13}$$

and, since $A_t^{\check{a}^\varepsilon} = A_t^a = 0$,

$$|\check{\Gamma}_t^\varepsilon - \Gamma_t|^2 = \check{U}_t^{\check{a}^\varepsilon} - U_t^{\Re,a}. \tag{4.14}$$

Combining (4.12), (4.13) and (4.14) with (4.10) and (4.11), we get

$$e^{\beta t}|(\check{Y}_t)^\ell - (\tilde{Y}_t^{\Re})^\ell|^2 \leqslant 2e^{\beta t}|\check{U}_t^{\check{a}} - \check{U}_t^{\check{a}^\varepsilon}|^2 + 2e^{\beta t}|\check{U}_t^{\check{a}} - U_t^{\Re,a}|^2$$
$$\leqslant C_\beta|\check{U}_t^{\check{a}} - \check{U}_t^{\check{a}^\varepsilon}|^2 + \mathbb{E}_t\Big[C_\beta\mathcal{E}(\Re) + 2\int_t^T e^{\beta u}|Y_u - \tilde{Y}_u^{\Re}|^2\mathrm{d}u\Big], \quad (4.15)$$

with

$$\mathcal{E}(\Re) := |N^{\check{a}}|^2 \sup_{s\in[0,T]}(1 + |X_s|^2 + |Y_s|^2 + |\tilde{Y}_s^{\Re}|^2)|\Re| + |N^{\check{a}}|^2 \sup_{1\leqslant k\leqslant\kappa}\sup_{r\in[r_{k-1},r_k]}|X_r - X_{r_k}|^2.$$

Importantly, the constant C_β does not depend on ε. Now, let us study the term $|\check{U}_t^{\check{a}} - \check{U}_t^{\check{a}^\varepsilon}|^2$. We recall that, for all $s \in [t, \theta_1^\varepsilon[$,

$$\check{U}_s^{\check{a}^\varepsilon} = \check{U}_{\theta_1^\varepsilon}^{\check{a}^\varepsilon} + \int_s^{\theta_1^\varepsilon} \check{f}^\ell(u, \check{V}_u^{\check{a}^\varepsilon})du - \int_s^{\theta_1^\varepsilon} \check{V}_u^{\check{a}^\varepsilon} dW_u - c^{\ell\alpha_1^\varepsilon}(X_{\theta_1^\varepsilon}) - c^{\alpha_1^\varepsilon\alpha_2^\varepsilon}(X_{\theta_1^\varepsilon})1_{\theta_1^\varepsilon=\theta_2^\varepsilon}$$

and

$$\check{U}_s^{\check{a}} = \check{Y}_s^\ell = \check{Y}_{\theta_1^\varepsilon}^\ell + \int_s^{\theta_1^\varepsilon} \check{f}^\ell(u, \check{Z}_u^\ell)du - \int_s^{\theta_1^\varepsilon} \check{Z}_u^\ell dW_u + \check{K}_{\theta_1^\varepsilon}^\ell - \check{K}_s^\ell.$$

Then, a straightforward adaptation of a classical stability result for BSDEs (see e.g. [10]) gives us

$$\mathbb{E}\big[|\check{U}_t^{\check{a}} - \check{U}_t^{\check{a}^\varepsilon}|^2\big] \leqslant C\mathbb{E}\Big[|\check{U}_{\theta_1^\varepsilon}^{\check{a}^\varepsilon} - \check{Y}_{\theta_1^\varepsilon}^\ell - c^{\ell\alpha_1^\varepsilon}(X_{\theta_1^\varepsilon}) - c^{\alpha_1^\varepsilon\alpha_2^\varepsilon}(X_{\theta_1^\varepsilon})1_{\theta_1^\varepsilon=\theta_2^\varepsilon}|^2 + |\check{K}_{\theta_1^\varepsilon}^\ell - \check{K}_t^\ell|^2\Big]$$

with a constant C that does not depend on ε. We can remark that

$$\check{U}_{\theta_1^\varepsilon}^{\check{a}^\varepsilon} = \check{U}_{\theta_1^\varepsilon}^{\check{a}} = \check{Y}_{\theta_1^\varepsilon}^{\alpha_1^\varepsilon}1_{\theta_1^\varepsilon\neq\theta_2^\varepsilon} + \check{Y}_{\theta_1^\varepsilon}^{\alpha_2^\varepsilon}1_{\theta_1^\varepsilon=\theta_2^\varepsilon},$$

and

$$[\check{Y}_{\theta_1^\varepsilon}^{\alpha_2^\varepsilon} - \check{Y}_{\theta_1^\varepsilon}^{\alpha_1^\varepsilon} - c^{\alpha_1^\varepsilon\alpha_2^\varepsilon}(X_{\theta_1^\varepsilon})]1_{\theta_1^\varepsilon=\theta_2^\varepsilon} = 0, \qquad [\check{Y}_{\theta_1^\varepsilon}^{\alpha_1^\varepsilon} - \check{Y}_{\theta_1^\varepsilon}^\ell - c^{\ell\alpha_1^\varepsilon}(X_{\theta_1^\varepsilon})]1_{\check{\theta}_1>t} = 0.$$

Thus we obtain

$$\mathbb{E}\big[|\check{U}_t^{\check{a}} - \check{U}_t^{\check{a}^\varepsilon}|^2\big] \leqslant C\mathbb{E}\Big[|\check{Y}_{\theta_1^\varepsilon}^{\alpha_1^\varepsilon} - \check{Y}_{\theta_1^\varepsilon}^\ell - c^{\ell\alpha_1^\varepsilon}(X_{\theta_1^\varepsilon})|^2 1_{\check{\theta}_1=t} + |\check{K}_{\theta_1^\varepsilon}^\ell - \check{K}_t^\ell|^2\Big].$$

Since $\theta_1^\varepsilon \xrightarrow{\varepsilon\to 0} \check{\theta}_1$ and $\check{Y}$, $\check{K}$ and X are continuous processes, by a direct domination we can apply the dominated convergence theorem to get

$$\lim_{\varepsilon\to 0}\mathbb{E}\big[|\check{U}_t^{\check{a}} - \check{U}_t^{\check{a}^\varepsilon}|^2\big] = 0.$$

Thus, when ϵ tends to 0 in (4.15), up to a subsequence, we obtain

$$e^{\beta t}|(\check{Y}_t)^\ell - (\tilde{Y}_t^{\Re})^\ell|^2 \leqslant \mathbb{E}_t\Big[C_\beta\mathcal{E}(\Re) + 2\int_t^T e^{\beta u}|Y_u - \tilde{Y}_u^{\Re}|^2\mathrm{d}u\Big]. \quad (4.16)$$

We now treat the case $t \in \Re$. This case is simpler than the case $t \notin \Re$ since we do not have to introduce an auxiliary strategy $\check{a}^\varepsilon$: we can do previous calculations directly with the optimal strategy $\check{a}$. The only difference comes from the fact that $A_t^{\check{a}}$ and A_t^a are not necessarily equal to 0, but, since $t \in \Re$, we have $A_t^{\check{a}} = A_t^a$ and so $A_t^{\check{a}} - A_t^a = 0$. Finally, (4.16) stays true when $t \in \Re$.

2.c Combining (4.7) and (4.16) with (4.6), we obtain, for all $t \leqslant s \leqslant T$,

$$\mathbb{E}_t\Big[e^{\beta s}|Y_s - \tilde{Y}_s^{\Re}|^2\Big] \leqslant C_\beta \mathbb{E}_t[\mathcal{E}(\Re)] + 2\int_s^T \mathbb{E}_t\Big[e^{\beta u}|Y_u - \tilde{Y}_u^{\Re}|^2\Big]\,du.$$

Then, a direct application of Gronwall lemma gives us

$$|Y_t - \tilde{Y}_t^{\Re}|^2 \leqslant \mathbb{E}_t\Big[e^{\beta t}|Y_t - \tilde{Y}_t^{\Re}|^2\Big] \leqslant C_\beta \mathbb{E}_t[\mathcal{E}(\Re)]\,.$$

Using Jensen inequality, Doob maximal inequality and Cauchy-Schwarz inequality, the previous inequality allows us to obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t - \tilde{Y}_t^{\Re}|^2\right] \leqslant & C\mathbb{E}\big[\mathcal{E}(\Re)^2\big]^{1/2} \\ \leqslant & C\mathbb{E}\big[|N^{\check{a}}|^8\big]^{1/4}\mathbb{E}\left[\sup_{s\in[0,T]}(1+|X_s|^8+|Y_s|^8+|\tilde{Y}_s^{\Re}|^8)\right]^{1/4}|\Re| \\ & + C\mathbb{E}\big[|N^{\check{a}}|^8\big]^{1/4}\mathbb{E}\left[\sup_{1\leqslant k\leqslant\kappa}\sup_{r\in[r_{k-1},r_k]}|X_r - X_{r_k}|^8\right]^{1/4}.\end{aligned}$$

Finally, we just have to apply estimates of Proposition 4.1, Proposition 2.3, classical estimate for X, and Theorem 1 in [12] to get

$$\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t - \tilde{Y}_t^{\Re}|^2\right] \leqslant C|\Re| + C|\Re|\log(2T/|\Re|)$$

and

$$\mathbb{E}\left[\sup_{r\in\Re}|Y_r - Y_r^{\Re}|^2\right] \leqslant \mathbb{E}\left[\sup_{t\in[0,T]}|Y_{t^-} - \tilde{Y}_{t^-}^{\Re}|^2\right] \leqslant C|\Re| + C|\Re|\log(2T/|\Re|).$$

To conclude the proof, we just have to remark that the term $\sup_{1\leqslant k\leqslant\kappa}\sup_{r\in[r_{k-1},r_k]}|X_r - X_{r_k}|^2$ does not appear in $\mathcal{E}(\Re)$ when cost functions are constant. □

Proposition 4.3. *Let us assume that (Hz) and (Hf) are in force, then the following holds:*

$$\mathbb{E}\left[\int_0^T \left|Z_s - Z_s^{\Re}\right|^2 ds\right] \leqslant C\,|\Re|^{1/2}\sqrt{\log(2T/|\Re|)}.$$

If cost functions are constant, previous inequality holds true without the term $\sqrt{\log(2T/|\Re|)}$.

Proof. Introduce $\delta\widetilde{Y} := Y - \widetilde{Y}^{\Re}$, $\delta Y := Y - Y^{\Re}$, $\delta Z := Z - Z^{\Re}$ and $\delta f := f(X,Y,Z) - f(X, \widetilde{Y}^{\Re}, Z^{\Re})$. Applying Itô's formula to the *càdlàg* process $|\delta\widetilde{Y}|^2$, we get

$$|\delta\widetilde{Y}_0|^2 + \int_0^T |\delta Z_s|^2 ds = |\delta\widetilde{Y}_T|^2 - 2\int_{(0,T]} \delta\widetilde{Y}_{s^-} d\delta\widetilde{Y}_s - \sum_{0<s\leqslant T} |\delta\widetilde{Y}_s - \delta Y_s|^2.$$

Recalling that $\delta\widetilde{Y}_{s^-} = \delta Y_s$, $\int_{(0,T]} \delta Y_s dK_s^{\Re} \geqslant 0$ and the Lipschitz property of f, standard arguments lead to

$$\mathbb{E}\Big[|\delta\widetilde{Y}_0|^2\Big] + \mathbb{E}\Big[\int_0^T |\delta Z_s|^2 ds\Big] \leqslant C\mathbb{E}\Big[\int_0^T \delta Y_s dK_s\Big] \leqslant C\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |\delta Y_t|^2\Big]^{1/2} \mathbb{E}\big[K_T^2\big]^{1/2}.$$

Then, using Proposition 4.1 and Proposition 4.2 concludes the proof. □

As a by-product we get a strong estimate on Z.

Corollary 4.1. *Let us assume that assumption (Hf) is in force. Then we have*

$$|Z_t| \leqslant \bar{L}(1 + |X_t|) \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}s \ a.e.$$

where $\bar{L}$ is the constant that appears in (2.20).

Proof. Let us introduce a new generator $\hat{f}(x,y,z) := f(x,y,\rho_x(z))$ with ρ_x the projection on the Euclidean ball of radius $\bar{L}(1+|x|)$ where $\bar{L}$ comes from the estimate on $Z^{\Re}$ given in (2.20). We easily have that $\hat{f}$ is a Lipschitz function such that

$$\Big|\hat{f}(x,y,z)\Big| \leqslant C(1 + |x| + |y|).$$

We denote $(\hat{Y}, \hat{Z}, \hat{K})$ the solution of the obliquely reflected BSDE with generator $\hat{f}$. Since (Hf) is in force, we can use (2.20) for the discretely reflected BSDE with generator $\hat{f}$ and we get that

$$|\hat{Z}_t^{\Re}| \leqslant \bar{L}(1 + |X_t|) \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}s \text{ a.e.}$$

Using Proposition 4.3, we take $|\Re| \to 0$ and we obtain that

$$|\hat{Z}_t| \leqslant \bar{L}(1 + |X_t|) \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}s \text{ a.e.}$$

and then

$$\hat{f}(s, X_s, \hat{Y}_s, \hat{Z}_s) = f(s, X_s, \hat{Y}_s, \hat{Z}_s) \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}s \text{ a.e.}$$

Thus, by uniqueness of the solution to the obliquely reflected BSDE, we have that $\hat{Z} = Z$, concluding the proof of the Corollary. □

Theorem 4.1. *We assume that (Hf) is in force. Then we have*

$$\mathbb{E}\left[\sup_{r\in\Re} |Y_r - Y_r^{\Re}|^2 + \sup_{t\in[0,T]} |Y_t - \tilde{Y}_t^{\Re}|^2\right] \leqslant C|\Re| \log(2T/|\Re|), \tag{4.17}$$

and

$$\mathbb{E}\left[\int_0^T \left|Z_s - Z_s^{\Re}\right|^2 ds\right] \leqslant C\sqrt{|\Re|\log(2T/|\Re|)}. \tag{4.18}$$

If, furthermore, cost functions are constant, previous estimates hold true without the $\log(2T/|\Re|)$ *term.*

Proof. Thanks to Corollary 4.1, we can replace the generator f by $\hat{f}(x,y,z) := f(x,y,\rho_x(z))$ with ρ_x the projection on the Euclidean ball of radius $\bar{L}(1+|x|)$ without modifying our BSDEs. Since (Hz) is in force for the generator $\hat{f}$, we can apply Proposition 4.2 and Proposition 4.3 and the theorem is proved. □

4.2 Proof of Theorem 1.1

Combining the previous results with the control of the error between the discrete-time scheme and the discretely obliquely reflected BSDE derived in Section 3, we obtain the convergence of the discrete time scheme to the solution of the continuously obliquely reflected BSDE. Namely, we just have to put together Theorem 4.1 and Theorem 3.1.

A Appendix

A.1 Proof of Proposition 2.3

Observing that on each interval $[r_j, r_{j+1})$, $(\tilde{Y}^{\Re}, Z^{\Re})$ solves a standard BSDE, existence and uniqueness follow from a concatenation procedure and [21].

Concerning estimates, we cannot apply directly Proposition 2.1 in [8] since we have a generator f with a coupling in y. Our strategy is to apply Proposition 2.1 with terminal condition $\xi = g(X_T)$, random generator $F(s,z) = f(X_s, \tilde{Y}_s^{\Re}, z)$ and costs $C_s^{ij} = c^{ij}(X_s)$. So, we just have to show that (HF_p) is in force. Thus, using the fact that f is a Lipschitz function with respect to y, it is sufficient to control $\widetilde{Y}^{\Re}$ in $\mathscr{S}^p$.

As in the proof of Theorem 2.4 in [16], we consider two nonreflected BSDEs bounding $\widetilde{Y}^{\Re}$. Define the $\mathbb{R}^d$-valued random variable $\breve{g}(X_T)$ and the random map $\breve{f}$ by $(\breve{g})^j(x) := \sum_{i=1}^d \left|(g)^i\right|$ and $(\breve{f})^j(\omega,t,z) := \sum_{i=1}^d \left|(f)^i(X_t(\omega), \widetilde{Y}_t^{\Re}(\omega), z)\right|$ for $1 \leqslant j \leqslant d$. We then denote by $(\breve{Y}, \breve{Z}) \in (\mathscr{S}^p \times \mathscr{H}^p)$ the solution of the following nonreflected BSDE:

$$\breve{Y}_t = \breve{g}(X_T) + \int_t^T \breve{f}(s, \breve{Z}_s)ds - \int_t^T \breve{Z}_s dW_s, \quad 0 \leqslant t \leqslant T.$$

Since all the components of $\breve{Y}$ are similar, $\breve{Y} \in \mathcal{Q}$. We also introduce $(\mathring{Y}, \mathring{Z})$ the solution of the BSDE

$$\mathring{Y}_t = g(X_T) + \int_t^T f(X_s, \widetilde{Y}_s^{\Re}, \mathring{Z}_s)ds - \int_t^T \mathring{Z}_s dW_s, \quad 0 \leqslant t \leqslant T.$$

Using a comparison argument on each interval $[r_j, r_{j+1})$ and the monotonicity property of $\mathcal{P}$, we straightforwardly deduce $(\mathring{Y})^i \leqslant (\widetilde{Y}^{\Re})^i \leqslant (\breve{Y})^i$, for all $1 \leqslant i \leqslant d$. Since $(\mathring{Y}, \breve{Y})$

are solutions to standard non-reflected BSDEs, classical estimates (see e.g. [2]) lead to

$$
\begin{aligned}
\mathbb{E}_t\Big[\sup_{t\leqslant s\leqslant T}|\widetilde{Y}^{\mathfrak{R}}_s|^p\Big] &\leqslant \mathbb{E}_t\Big[\sup_{t\leqslant s\leqslant T}|\mathring{Y}_s|^p+\sup_{t\leqslant s\leqslant T}|\breve{Y}_s|^p\Big]\\
&\leqslant C_p\mathbb{E}_t\left[|g(X_T)|^p+\int_t^T\left|f(X_s,\widetilde{Y}^{\mathfrak{R}}_s,0)\right|^p ds\right]\\
&\leqslant C_p\mathbb{E}_t\left[1+\sup_{s\in[t,T]}|X_s|^p+\int_t^T\sup_{s\leqslant u\leqslant T}\left|\widetilde{Y}^{\mathfrak{R}}_u\right|^p ds\right].
\end{aligned}
$$

Finally, using Gronwall lemma we get

$$
\mathbb{E}_t\Big[\sup_{t\leqslant s\leqslant T}|\widetilde{Y}^{\mathfrak{R}}_s|^p\Big]\leqslant C_p\,\mathbb{E}_t\left[1+\sup_{s\in[t,T]}|X_s|^p\right]
$$

which leads to, recall (2.7),

$$
\mathbb{E}_t\Big[\sup_{t\leqslant s\leqslant T}|\widetilde{Y}^{\mathfrak{R}}_s|^p\Big]\leqslant C_p\,(1+|X_t|^p)\,, \tag{A.1}
$$

and in particular to $|\tilde{Y}^{\mathfrak{R}}|_{\mathscr{S}_p}\leqslant C_p$.

A.2 A priori estimates

In this section, we prove a generic estimate for a process that can be represented by using switched BSDEs. This result is tailor-made for the solution of obliquely reflected BSDEs. For a positive process $\beta\in\mathscr{S}^2$, we denote by $\bar{\mathscr{A}}$ the set of strategies $a\in\mathscr{A}$, satisfying

$$
\mathbb{E}_t\big[|N^a|^2\big]^{\frac{1}{2}}\leqslant\beta_t\,,\qquad\text{for } t\leqslant T. \tag{A.2}
$$

We consider a process $\mathfrak{X}\in\mathscr{S}^p$, for all $p\geqslant 2$, and for $a\in\bar{\mathscr{A}}$, we define

$$
\mathfrak{A}^a_t\ :=\ \sum_{j=1}^{N^a}\gamma^a_{\theta_j}\mathfrak{X}_{\theta_j}\mathbf{1}_{\{\theta_j\leqslant t\leqslant T\}}\ ,
$$

where γ is a process in $\mathscr{S}^2$ essentially bounded by a constant Λ. We also consider a process $\mathfrak{Y}\in\mathscr{S}^2$ which is given by $\mathfrak{Y}_t=(\mathfrak{Y}^i_t)_{1\leqslant i\leqslant d}$ s.t. $\mathfrak{Y}^i_t=\mathfrak{U}^a_t$ for some $a\in\bar{\mathscr{A}}\cap\mathscr{A}_{i,t}$ where, for $t\leqslant r\leqslant T$,

$$
\mathfrak{U}^a_r=\nu^a\mathfrak{X}_T+\int_t^T F^a(s,\mathfrak{X}_s,\mathfrak{U}^a_s,\mathfrak{V}^a_s,\mathfrak{Y}_s)\mathrm{d}s-\int_t^T\mathfrak{V}^a_s\mathrm{d}W_s+\mathfrak{A}^a_T-\mathfrak{A}^a_t\ .
$$

with ν^a a $\mathcal{F}_T$-measurable random variable essentially bounded by Λ and F a progressively measurable map satisfying

$$
|F^a(s,x,u,v,y)|\leqslant\Lambda(|x|+|u|+|v|+|y|)\ . \tag{A.3}
$$

Proposition A.1.

$$|\mathfrak{Y}_r|^2 \leqslant C_\Lambda(1+\beta_r)\mathbb{E}_r\left[\sup_{r\leqslant s\leqslant T}|\mathfrak{X}_s|^4\right]^{\frac{1}{2}}, \quad r\in[0,T].$$

Proof. Let us introduce $\mathfrak{G}^a = \mathfrak{U}^a + \mathfrak{A}^a$. Applying Ito's formula, we obtain for all $r \leqslant t \leqslant u \leqslant T$,

$$\mathbb{E}_r\left[|\mathfrak{G}^a_u|^2 + \int_u^T |\mathfrak{V}^a_s|^2 \mathrm{d}s\right] \leqslant \mathbb{E}_r\left[|\mathfrak{G}^a_T|^2 + 2\int_u^T \mathfrak{G}^a_s F^a(s,\mathfrak{X}_s,\mathfrak{U}^a_s,\mathfrak{V}^a_s,\mathfrak{Y}_s)\mathrm{d}s\right].$$

Using classical arguments and the assumption on F, we obtain

$$\mathbb{E}_r\big[|\mathfrak{G}^a_u|^2\big] \leqslant C_\Lambda \mathbb{E}_r\left[\sup_{t\leqslant s\leqslant T}|\mathfrak{X}_s|^2 + \int_u^T |\mathfrak{Y}_s|^2\mathrm{d}s\right] + \sup_{t\leqslant s\leqslant T}\mathbb{E}_r\big[|\mathfrak{A}^a_s|^2\big]. \tag{A.4}$$

We observe that, for $t \leqslant s \leqslant T$,

$$\begin{aligned}\mathbb{E}_r\big[|\mathfrak{A}^a_s|^2\big] &= \mathbb{E}_r\left[\left|\sum_{j=1}^{N^a}\gamma^a_{\theta_j}\mathfrak{X}_{\theta_j}\mathbf{1}_{\{\theta_j\leqslant s\leqslant T\}}\right|^2\right]\\ &\leqslant \Lambda\mathbb{E}_r\left[N^a\sup_{t\leqslant s\leqslant T}|\mathfrak{X}_s|^2\right] \leqslant \Lambda\beta_r\mathbb{E}_r\left[\sup_{t\leqslant s\leqslant T}|\mathfrak{X}_s|^4\right]^{\frac{1}{2}}.\end{aligned}$$

Inserting the previous inequality into (A.4), we obtain,

$$\mathbb{E}_r\big[|\mathfrak{G}^a_u|^2\big] \leqslant C_\Lambda(1+\beta_r)\mathbb{E}_r\left[\sup_{r\leqslant s\leqslant T}|\mathfrak{X}_s|^4\right]^{\frac{1}{2}} + C_\Lambda\mathbb{E}_r\left[\int_u^T|\mathfrak{Y}_s|^2\mathrm{d}s\right]. \tag{A.5}$$

In particular, for all $r \leqslant t \leqslant T$, we compute

$$\mathbb{E}_r\big[|\mathfrak{Y}_t|^2\big] = \sum_{i=1}^2\mathbb{E}_r\big[|\mathfrak{Y}^i_t|^2\big] \leqslant C_\Lambda(1+\beta_r)\mathbb{E}_r\left[\sup_{r\leqslant s\leqslant T}|\mathfrak{X}_s|^4\right]^{\frac{1}{2}} + C_\Lambda\mathbb{E}_r\left[\int_t^T|\mathfrak{Y}_s|^2\mathrm{d}s\right].$$

Using Gronwall Lemma, we get

$$|\mathfrak{Y}_r|^2 \leqslant C_\Lambda(1+\beta_r)\mathbb{E}_r\left[\sup_{r\leqslant s\leqslant T}|\mathfrak{X}_s|^4\right]^{\frac{1}{2}}.$$

□

A.3 Proof of Proposition 2.6

Before starting the proof, let us state the following estimates on the Λ-process appearing in the representation (2.18).

$$\sup_{a\in\mathscr{A}^{\Re}}\|\sup_{t\leqslant s\leqslant T}\Lambda^a_{t,s}\|_{\mathscr{L}^p} \leqslant C^p_L, \quad 0\leqslant t\leqslant T, \quad p\geqslant 2. \tag{A.6}$$

We also compute from the dynamics of Λ that

$$\sup_{a\in\mathscr{A}^{\Re}}\left(\|\Lambda^a_{t,t}-\Lambda^a_{t,u}\|_{\mathscr{L}^p}+\|\sup_{t\leqslant s\leqslant T}|\Lambda^a_{u,s}-\Lambda^a_{t,s}|\,\|_{\mathscr{L}^p}\right)\leqslant C^p_L\sqrt{t-u}\,,\quad u\leqslant t\leqslant T\,,\quad p\geqslant 2\,. \tag{A.7}$$

The proof of Proposition 2.6 follows from the same arguments as in the proof of Theorem 3.1 in [8]. The novelty comes from the term DY but the estimates (2.12)-(2.13) allow to control it without any difficulty. From Remark 2.5, it is clear that

$$\mathbb{E}\Big[\int_0^T|Z^{\Re}_s-\bar{Z}^{\Re}_s|^2\mathrm{d}s\Big]\leqslant\mathbb{E}\Big[\int_0^T|Z^{\Re}_s-Z^{\Re}_{\pi(s)}|^2\mathrm{d}s\Big]\,. \tag{A.8}$$

For $s\leqslant T$ and $a=(\alpha_k,\theta_k)_{k\geqslant 0}\in\mathscr{A}^{\Re}_{s,\ell}$ with $\ell\in\mathcal{I}$, we define $(V^a_{s,t})_{s\leqslant t\leqslant T}$ by

$$\begin{aligned}V^a_{s,t}:=\mathbb{E}_t\Big[&\partial_x g^{a_T}(X_T)\Lambda^a_{s,T}D_sX_T-\sum_{k=1}^{N^a}\partial_x c^{\alpha_{j-1},\alpha_j}(X_{\theta_k})\Lambda^a_{s,\theta_k}D_sX_{\theta_k}\\&+\int_s^T\Big(\partial_x f^{a_u}(\Theta^{\Re}_u)\Lambda^a_{s,u}D_sX_u+\partial_y f^a(\Theta^{\Re}_u)\Lambda^a_{s,u}D_s\widetilde{Y}^{\Re}_u)\Big)\,\mathrm{d}u\Big]\,.\end{aligned}$$

We now fix $\ell\in\mathcal{I}$ and denote by $a^u\in\mathscr{A}^{\Re}_{u,\ell}$, for $u\leqslant T$, the optimal strategy associated to the representation of $(\widetilde{Y}^{\Re}_u)^\ell$, recalling (ii) in Corollary 2.1.

Observe that, by definition, we have

$$N^{a^t}=N^{a^u}\text{ and }a^t=a^u,\text{ for all}\quad r_j\leqslant t\leqslant u<r_{j+1}\quad\text{and}\quad j<\kappa\,. \tag{A.9}$$

Fix $i<n$, and deduce from (2.18) and (A.9) that

$$\mathbb{E}\Big[|(Z^{\Re}_t)^\ell-(Z^{\Re}_{t_i})^\ell|^2\Big]=\mathbb{E}\Big[|V^{a^t}_{t,t}-V^{a^{t_i}}_{t_i,t_i}|^2\Big]\;\leqslant\;2\left(\mathbb{E}\Big[|V^{a^{t_i}}_{t,t}-V^{a^{t_i}}_{t_i,t}|^2\Big]+\mathbb{E}\Big[|V^{a^{t_i}}_{t_i,t}-V^{a^{t_i}}_{t_i,t_i}|^2\Big]\right)\,, \tag{A.10}$$

for $t\in[t_i,t_{i+1})$. Combining (Hr), (2.9), (2.10), (A.6), (A.7) and Cauchy-Schwartz inequality with the definition of V^a, we deduce

$$\mathbb{E}\Big[|V^{a^{t_i}}_{t,t}-V^{a^{t_i}}_{t_i,t}|^2\Big]\leqslant C_L|\pi|^{\frac{1}{2}}\,,\quad t_i\leqslant t\leqslant t_{i+1}\,,\quad i\leqslant n\,. \tag{A.11}$$

Since $V^{a^{t_i}}_{t_i,\cdot}$ is a martingale on $[t_i,t_{i+1}]$, we obtain

$$\begin{aligned}\mathbb{E}\Big[|V^{a^{t_i}}_{t_i,t}-V^{a^{t_i}}_{t_i,t_i}|^2\Big]&\leqslant\mathbb{E}\Big[|V^{a^{t_i}}_{t_i,t_{i+1}}-V^{a^{t_i}}_{t_i,t_i}|^2\Big]\\&\leqslant\mathbb{E}\Big[|V^{a^{t_i}}_{t_{i+1},t_{i+1}}|^2-|V^{a^{t_i}}_{t_i,t_i}|^2\Big]+\mathbb{E}\Big[|V^{a^{t_i}}_{t_i,t_{i+1}}|^2-|V^{a^{t_i}}_{t_{i+1},t_{i+1}}|^2\Big]\\&\leqslant\mathbb{E}\Big[|V^{a^{t_i}}_{t_{i+1},t_{i+1}}|^2-|V^{a^{t_i}}_{t_i,t_i}|^2\Big]+C_L|\pi|^{\frac{1}{2}}\,,\qquad t_i\leqslant t\leqslant t_{i+1}\,,\end{aligned} \tag{A.12}$$

where the last inequality follows from (A.11). Combining (A.10), (A.11), (A.12) and summing up over i, we obtain

$$\mathbb{E}\Big[\int_0^T |(Z_t^{\Re})^\ell - (Z_{\pi(t)}^{\Re})^\ell|^2 \mathrm{d}t\Big] \leqslant C_L|\pi|^{\frac{1}{2}} + |\pi|\Big(\mathbb{E}\Big[|V_{T,T}^{a^{r_{\kappa-1}}}|^2 - |V_{0,0}^{a^0}|^2\Big] + \sum_{j=1}^{\kappa-1}(|V_{r_j,r_j}^{a^{r_{j-1}}}|^2 - |V_{r_j,r_j}^{a^{r_j}}|^2)\Big).$$

Combined with (2.9) and (A.6), this concludes the proof since ℓ is arbitrary.

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